Systems of Kowalevski type, discriminantly separable polynomials and quad-graphs

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Abstract

We establish a new class of integrable *systems of Kowalevski type*, associated with discriminantly separable polynomials of degree two in each of three variables. Defining property of such polynomials, that all discriminants as polynomials of two variables are factorized as products of two polynomials of one variable each (denote one of the polynomial components as P), lead to an effective integration procedure. In the motivating example, the celebrated Kowalevski top, the discriminant separability is a property of the polynomial defining the Kowalevski fundamental equation. We construct several new examples of systems of Kowalevski type, and we perform their explicit integration in genus two theta-functions. One of the main tasks of the paper is to classify such discriminantly separable polynomials. Our classification is based on the study of structures of zeros of a polynomial component P of a discriminant.

From a geometric point of view, such a classification is related to the types of pencils of conics. We construct also discrete integrable systems on quad-graphs associated with discriminantly separable polynomials. We establish a relationship between our classification and the classification of integrable quad-graphs which has been suggested recently by Adler, Bobenko and Suris. As a fit back, we get a geometric interpretation of their results in terms of pencils of conics, and in the case of general position, when all four zeros of the polynomial P are distinct, we get a connection with the Buchstaber-Novikov two-valued groups on $\mathbb{CP}^1$.

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Contents

1 Introduction

1.1 Discriminantly separable polynomials - an overview

The Kowalevski top [24] is one of the most celebrated integrable systems. There is a waste literature dedicated to understanding of Kowalevski original integration procedure, to its modern versions and hidden symmetries (see for example [25], [23], [18], [26], [3], [13], [20], [17], [4], [6]).

In a very recent paper [15] of one of the authors of the present paper, a new approach to the Kowalevski integration procedure has been suggested. It has been based on a new notion introduced therein of *discriminantly separable polynomials*. A family of such polynomials has been constructed there as pencil equations from the theory of conics

$$\mathcal{F}(w, x_1, x_2) = 0,$$

where w, x_1, x_2 are the pencil parameter and the Darboux coordinates respectively. (For classical applications of the Darboux coordinates see Darboux's book [12], for modern applications see the book [16] and [14].) The key algebraic property of the pencil equation, as quadratic equation in each of three variables w, x_1, x_2 is: *all three of its discriminants are expressed as products of two polynomials in one variable each*:

$$\begin{aligned} \mathcal{D}_w(\mathcal{F})(x_1, x_2) &= P(x_1)P(x_2) \\ \mathcal{D}_{x_1}(\mathcal{F})(w, x_2) &= J(w)P(x_2) \\ \mathcal{D}_{x_2}(\mathcal{F})(w, x_1) &= P(x_1)J(w) \end{aligned} \tag{1}$$

where J, P are polynomials of degree 3 and 4 respectively, and the elliptic curves

$$\Gamma_1 : y^2 = P(x), \quad \Gamma_2 : y^2 = J(s)$$

are isomorphic (see Proposition 1 of [15]) .

The so-called *fundamental Kowalevski equation* (see formula (11) below, and also [24], [23], [18])

$$Q(w, x_1, x_2) = 0,$$

appeared to be an example of a member of the family, as it was shown in [15] (Theorem 3). Moreover, all main steps of the Kowalevski integration now follow as easy and transparent logical consequences of the theory of discriminantly separable polynomials. Let us mention here just one relation, see Corollary 1 from [15] (known in the context of the Kowalevski top as *the Kowalevski magic change of variables*):

$$\begin{aligned} \frac{dx_1}{\sqrt{P(x_1)}} + \frac{dx_2}{\sqrt{P(x_2)}} &= \frac{dw_1}{\sqrt{J(w_1)}} \\ \frac{dx_1}{\sqrt{P(x_1)}} - \frac{dx_2}{\sqrt{P(x_2)}} &= \frac{dw_2}{\sqrt{J(w_2)}}. \end{aligned} \tag{2}$$

There are two natural and important questions in this context:

1) *Are there any other discriminantly separable polynomials of degree two in each of three variables, beside those constructed from pencils of conics? In addition, the question is to perform a classification of such polynomials.*

2) *Are there other integrable dynamical systems related to discriminantly separable polynomials?*

The main issue of this paper is to address these two key questions.

In order to make precise the first question, one needs to specify a gauge group or the classes of equivalence up to which a classification would be performed. This leads to the group of Möbius transformations, as introduced in Corollary 3 of [15]:

$$\begin{aligned} x_1 &\mapsto \frac{a_1x_1+b_1}{c_1x_1+d_1} \\ x_2 &\mapsto \frac{a_2x_2+b_2}{c_2x_2+d_2} \\ w &\mapsto \frac{a_3w+b_3}{c_3w+d_3}. \end{aligned} \tag{3}$$

The family of discriminantly separable polynomials in three variables of degree two in each of them, constructed from pencils of conics served as a motivation to introduce more general classes of **discriminantly separable polynomials**. Let us recall here the definitions from [15]: a polynomial $F(x_1,\dots,x_n)$ is *discriminantly separable* if there exist polynomials $f_i(x_i)$ such that for every $i=1,\dots,n$

$$\mathcal{D}_{x_i}F(x_1,\dots,\hat{x}_i,\dots,x_n)=\prod_{j\neq i}f_j(x_j).$$

It is *symmetrically discriminantly separable* if

$$f_2=f_3=\dots=f_n,$$

while it is *strongly discriminantly separable* if

$$f_1=f_2=f_3=\dots=f_n.$$

It is *weakly discriminantly separable* if there exist polynomials $f_i^j(x_i)$ such that for every $i=1,\dots,n$

$$\mathcal{D}_{x_i}F(x_1,\dots,\hat{x}_i,\dots,x_n)=\prod_{j\neq i}f_j^i(x_j).$$

The classification of strongly discriminantly separable polynomials $\mathcal{F}(x_1,x_2,x_3)$ of degree two in each of three variables modulo fractional-linear transformations from formulae (3) with

$$a_1=a_2=a_3,\ b_1=b_2=b_3,\ c_1=c_2=c_3,\ d_1=d_2=d_3,$$

as the gauge group, is one of the main tasks of the present paper.

This classification is heavily based on the classification of pencils of conics (see [16], for example, for more details about pencils of conics). In the case of general position, the conics of a pencil intersect in four distinct points, and we code such situation with $(1, 1, 1, 1)$, see Fig. 1. It corresponds to the case where polynomial P has four simple zeros. In this case, the family of strongly discriminantly separable polynomials coincides with the family constructed in [15] from a general pencil of conics. This family, as it has been indicated in [15], corresponds to the two-valued Buchstaber-Novikov group associated with a cubic curve

$$\Gamma_2 : y^2 = J(s).$$

Two-valued groups of the form (Γ_2, Z_2) have been introduced in [8]. The Kowalevski change of variables (19) appeared to be an infinitesimal of the two-valued operation in this group, see [15]. (The theory of n valued-groups originates from a pioneering paper [10]. For a modern account see [7], and for higher-genus analogues, see [9].)

Other cases within the classification with nonzero polynomial P correspond to the situation where:

B the polynomial P has two simple zeros and one double zero, we code it $(1, 1, 2)$, and the conics of the corresponding pencil intersect in two simple points, and they have a common tangent in the third point of intersection, see Fig. 2;

C the polynomial P has two double zeros, code $(2, 2)$, and the conics of the corresponding pencil intersect in two points, having a common tangent in each of the points of intersection, see Fig. 3;

D the polynomial P has one simple zero and one triple zero, $(1, 3)$, and the conics of the corresponding pencil intersect in one simple point, and they have another common point of tangency of third order, see Fig. 4;

E the polynomial P has one quadriple zero, (4), and the conics of the corresponding pencil intersect in one point, having tangency of fourth order there, see Fig. 5.

For more details, see Section 5.1.

Referring to the second key question, about constructing other integrable dynamical systems related to discriminantly separable polynomials, we address it in two main directions. First, we are constructing a new class of integrable continuous systems, generalizing the Kowalevski top. Thus we call the members of that class – *the systems of Kowalevski type*. The relationship with discriminantly separable polynomials gives us possibility to perform effective integration procedure, and to provide an explicit integration formulae in theta-functions, in general associated with genus two curves, as in the original case of Kowalevski. Let us point out here one very important moment regarding the Kowalevski top

and all the systems of Kowalevski type: the main issue in integration procedures is related to the elliptic curves Γ_1, Γ_2 and the two-valued groups related to these elliptic curves, although, as we know, the final part of integration of the Kowalevski is related to a genus two curve. This, in a sense unexpected, surprising or possibly not clarified enough jump in genus from 1 to 2 is now explained in our Theorem 1, and it becomes a trade mark of all systems of Kowalevski type.

The second line of our treatment of the second key question connects discriminantly separable polynomials of degree two in each of three variables with discrete integrable systems on quad-graphs. See [1] and [2] for more details about quad-graphs; some basic notions from there are collected in Section 5.2. For each discriminantly separable polynomial we construct explicit discrete integrable system in Section 5.2. Thus, we establish a natural correspondence between our classification of discriminantly separable polynomials of degree two in three variables, mentioned above and described in Section 5.1, and the classification of integrable quad-graphs performed in [2]. This correspondence gives, as a fit back benefit, a geometric interpretation of constructions of Adler, Bobenko and Suris in terms of pencils of conics. Developing further this correspondence, in the case coded $(1, 1, 1, 1)$, with a polynomial P with four distinct zeros, we come to the conclusion that such integrable quad-graphs are related to the two-valued Buchstaber-Novikov groups associated with elliptic curves, i.e. those defined on $\mathbb{CP}^1$.

1.2 An overview of the paper

The paper is organized as follows. In the Section 2 we introduce *systems of Kowalevski type* of differential equations by generalizing Kowalevski's considerations. In the second subsection we construct an example of such system and in the third subsection we give integration procedure in terms of functions P_i, P_{ij}. Final formulae in terms of genus two theta-functions will follow from the Appendix. Following generalized Kötter transformation from [15], here we reformulate it for polynomials of degree three. That gives us possibility to integrate our examples in two ways, using properties of $\wp$-function and using generalized Kötter transformation. The subsections 2.4 and 2.5 present two methods for obtaining systems of Kowalevski type by studying first integrals. The Kowalevski top may be seen as a special subcase, and it serves as a principle motivating example.

In the Section 3 we consider a simple deformation of Kowalevski case by using simplest linear gauge transformations and we get *the Jurdjevic elasticae* from [20]. Systems analogue to the Jurdjevic elasticae have been obtained before by Komarov and Kuznetsov (see [21] and [22]), and they also serve as motivating examples for our study of systems of Kowalevski type. We give here the explicit solutions of all these problems in terms of P_i, P_{ij} functions, and ultimately, in

genus-two theta-functions, see the Appendix. Another system of Kowalevski type is constructed in the Section 4 following [24] and by use of a sequence of skilful tricks and identities.

The Section 5 is devoted to the strongly discriminantly separable polynomials in three variables, of degree two in each oh them. We classify in Section 5.1 such polynomials modulo Möbius transformations, depending on the structure of roots of the polynomial P, a factor of the discriminant. It appears that if polynomial P has four simple zeros, strongly discriminantly separable polynomials are equivalent to two-valued groups, see [9], [7], [8]. In the Section 5.2 we explicitly construct a discrete integrable system on quad-graphs associated to each strongly discriminantly separable polynomial. This way, we establish a morphism between our classification of strongly discriminantly separable polynomials and the diagonal part of the classification table from the theory of integrable systems on quad-graphs of Adler, Bobenko and Suris, (see [1], [2] for basic facts about integrable the quad graphs). To the off-diagonal part of the classification table of Adler, Bobenko and Suris, we associate symmetrically discriminantly separable polynomials.

The last Section 6 links our classification form Section 5 and the methods for obtaining systems of Kowalevski type described in the Section 2.4 and 2.5. We give two more examples of systems that can be integrated in the way proposed in Section 2.3.

Finally, in the Appendix we give explicit formulae for expressing P_i, P_{ij} functions in terms of theta functions of genus two. We also show briefly the derivation of those formulae, following [24], [5]. A modern account of the theory of theta-functions one can find for example in [17] or [27].

Since the main part of this paper is motivated by the Kowalevski top and by the Kowalevski integration procedure, and since it is the milestone of the classical integrable systems, we find it useful to extract some of the key moments of the Kowalevski work [24], here.

1.3 Basic lines of the Kowalevski integration procedure

Let us recall briefly that the Kowalevski top [24] is a heavy spinning top rotating about a fixed point, under the conditions $I_1 = I_2 = 2I_3$, $I_3 = 1$, $y_0 = z_0 = 0$. Here (I_1, I_2, I_3) denote the principal moments of inertia, (x_0, y_0, z_0) is the center of mass, $c = Mgx_0$, M is the mass of the top, (p, q, r) is the vector of angular velocity and $(\gamma_1, \gamma_2, \gamma_3)$ are cosines of the angles between z-axis of the fixed coordinate system and the axes of the coordinate system that is attached to the top and whose origin coincides with the fixed point.

Then the equations of motion take the following form, see [24], [18]:

$$\begin{aligned}
2\dot{p} &= qr \\
2\dot{q} &= -pr - c\gamma_3 \\
\dot{r} &= c\gamma_2 \\
\dot{\gamma}_1 &= r\gamma_2 - q\gamma_3 \\
\dot{\gamma}_2 &= p\gamma_3 - r\gamma_1 \\
\dot{\gamma}_3 &= q\gamma_1 - p\gamma_2.
\end{aligned} \tag{4}$$

System (4) has three well known integrals of motion and a fourth integral discovered by Kowalevski

$$\begin{aligned}
&2(p^2 + q^2) + r^2 = 2c\gamma_1 + 6l_1 \\
&2(p\gamma_1 + q\gamma_2) + r\gamma_3 = 2l \\
&\gamma_1^2 + \gamma_2^2 + \gamma_3^2 = 1 \\
&\left((p + \imath q)^2 + \gamma_1 + \imath\gamma_2\right)\left((p - \imath q)^2 + \gamma_1 - \imath\gamma_2\right) = k^2.
\end{aligned} \tag{5}$$

A change of variables

$$\begin{aligned}
x_1 &= p + \imath q \\
x_2 &= p - \imath q \\
e_1 &= x_1^2 + c(\gamma_1 + \imath\gamma_2) \\
e_2 &= x_2^2 + c(\gamma_1 - \imath\gamma_2)
\end{aligned} \tag{6}$$

transforms the four first integrals (5) into

$$\begin{aligned}
r^2 &= E + e_1 + e_2 \\
rc\gamma_3 &= G - x_2e_1 - x_1e_2 \\
c^2\gamma_3^2 &= F + x_2^2e_1 + x_1^2e_2 \\
e_1e_2 &= k^2,
\end{aligned} \tag{7}$$

with

$$\begin{aligned}
E &= 6l_1 - (x_1 + x_2)^2 \\
F &= 2cl + x_1x_2(x_1 + x_2) \\
G &= c^2 - k^2 - x_1^2x_2^2.
\end{aligned}$$

From the first integrals, one gets

$$(E + e_1 + e_2)(F + x_2^2e_1 + x_1^2e_2) - (G - x_2e_1 - x_1e_2)^2 = 0$$

which can be rewritten in the form

$$e_1P(x_2) + e_2P(x_1) + R_1(x_1, x_2) + k^2(x_1 - x_2)^2 = 0 \tag{8}$$

where polynomial P is

$$P(x_i) = x_i^2 E + 2x_1 F + G = -x_i^4 + 6l_1 x_i^2 + 4lcx_i + c^2 - k^2,\ i = 1, 2$$

and

$$\begin{aligned} &R_1(x_1, x_2) = EG - F^2 \\ &= -6l_1 x_1^2 x_2^2 - (c^2 - k^2)(x_1 + x_2)^2 - 4lc(x_1 + x_2)x_1 x_2 + 6l_1(c^2 - k^2) - 4l^2 c^2. \end{aligned}$$

Note that P from the formula above depends only on one variable, which is not obvious from its definition. Denote

$$R(x_1, x_2) = Ex_1 x_2 + F(x_1 + x_2) + G.$$

From (8), Kowalevski gets

$$(\sqrt{P(x_1)e_2} \pm \sqrt{P(x_2)e_1})^2 = -(x_1 - x_2)^2 k^2 \pm 2k\sqrt{P(x_1)P(x_2)} - R_1(x_1, x_2). \quad (9)$$

After a few transformations, (9) can be written in the form

$$\left[\sqrt{e_1}\frac{\sqrt{P(x_2)}}{x_1 - x_2} \pm \sqrt{e_2}\frac{\sqrt{P(x_1)}}{x_1 - x_2}\right]^2 = (w_1 \pm k)(w_2 \mp k), \quad (10)$$

where w_1, w_2 are the solutions of an equation, quadratic in s:

$$Q(s, x_1, x_2) = (x_1 - x_2)^2 s^2 - 2R(x_1, x_2)s - R_1(x_1, x_2) = 0. \quad (11)$$

The quadratic equation (11) is known as **the Kowalevski fundamental equation**. The discriminant separability condition for $Q(s, x_1, x_2)$ is satisfied

$$\mathcal{D}_s(Q)(x_1, x_2) = 4P(x_1)P(x_2)$$

$$\mathcal{D}_{x_1}(Q)(s, x_2) = -8J(s)P(x_2),\ \mathcal{D}_{x_2}(Q)(s, x_1) = -8J(s)P(x_1)$$

with

$$J(s) = s^3 + 3l_1 s^2 + s(c^2 - k^2) + 3l_1(c^2 - k^2) - 2l^2 c^2.$$

The equations of motion (4) can be rewritten in new variables $(x_1, x_2, e_1, e_2, r, \gamma_3)$ in the form:

$$\begin{aligned} 2\dot{x}_1 &= -if_1 \\ 2\dot{x}_2 &= if_2 \\ \dot{e}_1 &= -me_1 \\ \dot{e}_2 &= me_2 \end{aligned} \quad (12)$$

with two additional differential equations for $\dot{r}$ and $\dot{\gamma}_3$, where $m = \imath r$ and

$$f_1 = rx_1 + c\gamma_3,\ f_2 = rx_2 + c\gamma_3.$$

One can easily check that

$$\begin{aligned} f_1^2 &= P(x_1) + e_1(x_1 - x_2)^2 \\ f_2^2 &= P(x_2) + e_2(x_1 - x_2)^2. \end{aligned} \tag{13}$$

Further integration procedure is described in [24], and in the Subsection 2.3, we are going to develop analogue techniques for more general systems in details.

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2 On new integrable systems of Kowalevski type

2.1 Systems of Kowalevski type. Defintion

Now, we are going to introduce a class of dynamical systems, which generalize the Kowalevski top. Instead of the Kowalevski fundamental equation (see formula (11)), we start here from an arbitrary discriminantly separable polynomial of degree two in each of three variables.

Given a discriminantly separable polynomial of the second degree in each of three variables

$$\mathcal{F}(x_1, x_2, s) := A(x_1, x_2)s^2 + B(x_1, x_2)s + C(x_1, x_2), \tag{14}$$

such that

$$\mathcal{D}_s(\mathcal{F})(x_1, x_2) = B^2 - 4AC = 4P(x_1)P(x_2),$$

and

$$\begin{aligned} \mathcal{D}_{x_1}(\mathcal{F})(s, x_2) &= 4P(x_2)J(s) \\ \mathcal{D}_{x_2}(\mathcal{F})(s, x_1) &= 4P(x_1)J(s). \end{aligned}$$

Suppose, that a given system in variables x_1, x_2, e_1, e_2, r, γ_3, after some transformations reduces to

$$
\begin{aligned}
2\dot{x}_1 &= -if_1 \\
2\dot{x}_2 &= if_2 \\
\dot{e}_1 &= -me_1 \\
\dot{e}_2 &= me_2
\end{aligned}
\tag{15}
$$

where

$$
\begin{aligned}
f_1^2 &= P(x_1) + e_1 A(x_1, x_2) \\
f_2^2 &= P(x_2) + e_2 A(x_1, x_2).
\end{aligned}
\tag{16}
$$

Suppose additionally, that the first integrals of the initial system reduce to a relation

$$
P(x_2)e_1 + P(x_1)e_2 = C(x_1, x_2) - e_1 e_2 A(x_1, x_2). \tag{17}
$$

The equations for $\dot{r}$ and $\dot{\gamma}_3$ are not specified for the moment and m is a function of system's variables.

If a system satisfies the above assumptions we will call it *a system of Kowalevski type*. As it has been pointed out in the Introduction, see formulae (8, 11, 12,13), the Kowalevski top is an example of the systems of Kowalevski type.

The following theorem is quite general, and concerns all the systems of the Kowalevski type. It explains in full a subtle mechanism of a quite miraculous jump in genus, from one to two, in integration procedure, which has been observed in the Kowalevski top, and now it is going to be established as a characteristic property of the whole new class of systems.

Theorem 1 *Given a system which reduces to (15, 16, 17). Then the system is linearized on the Jacobian of the curve*

$$
y^2 = J(z)(z-k)(z+k),
$$

where J is a polynomial factor of the discriminant of $\mathcal{F}$ as a polynomial in x_1 and k is a constant such that

$$
e_1 e_2 = k^2.
$$

Proof. Indeed, from the equations of motion on e_i we get

$$
e_1 e_2 = k^2,
$$

with some constant k. Now, we get

$$
\left(\sqrt{e_1}\sqrt{P(x_2)} \pm \sqrt{e_2}\sqrt{P(x_1)}\right)^2 = C(x_1, x_2) - k^2 A(x_1, x_2) \pm 2\sqrt{P(x_1)P(x_2)}k.
$$

From the last relations, we get

$$
\left(\sqrt{e_1}\sqrt{\frac{P(x_2)}{A}} + \sqrt{e_2}\sqrt{\frac{P(x_1)}{A}}\right)^2 = (s_1 - k)(s_2 + k)
$$

and

$$\left(\sqrt{e_1}\sqrt{\frac{P(x_2)}{A}}-\sqrt{e_2}\sqrt{\frac{P(x_1)}{A}}\right)^2=(s_1+k)(s_2-k),$$

where s_1, s_2 are the solutions of the quadratic equation

$$\mathcal{F}(x_1,x_2,s)=0$$

in s. From the last equations we get

$$\begin{aligned}
2\sqrt{e_1}\sqrt{\frac{P(x_2)}{A}} &= \sqrt{(s_1-k)(s_2+k)}+\sqrt{(s_1+k)(s_2-k)}\\
2\sqrt{e_2}\sqrt{\frac{P(x_1)}{A}} &= \sqrt{(s_1-k)(s_2+k)}-\sqrt{(s_1+k)(s_2-k)}.
\end{aligned}$$

Since s_i are solutions of the quadratic equation $F(x_1,x_2,s_i)=0$, using Viète formulae and discriminant separability condition, we get

$$\begin{aligned}
s_1+s_2 &= -\frac{B}{A}\\
-(s_1-s_2) &= \frac{\sqrt{4P(x_1)P(x_2)}}{A}.
\end{aligned} \tag{18}$$

From the last equation, we get

$$(s_1-s_2)^2=4\frac{P(x_1)P(x_2)}{A^2}.$$

Using the last equation, we have

$$\begin{aligned}
f_1^2 &= \frac{P(x_1)}{(s_1-s_2)^2}\left[(s_1-s_2)^2+4e_1\frac{P(x_2)}{A^2}A\right]\\
&= \frac{P(x_1)}{(s_1-s_2)^2}\left[(s_1-s_2)^2+\left(\sqrt{(s_1-k)(s_2+k)}+\sqrt{(s_1+k)(s_2-k)}\right)^2\right]\\
&= \frac{P(x_1)}{(s_1-s_2)^2}\left[\sqrt{(s_1-k)(s_1+k)}+\sqrt{(s_2+k)(s_2-k)}\right]^2.
\end{aligned}$$

Similarly

$$f_2^2=\frac{P(x_2)}{(s_1-s_2)^2}\left[\sqrt{(s_1-k)(s_1+k)}-\sqrt{(s_2+k)(s_2-k)}\right]^2.$$

From the last two equations and from the equations of motion, we get

$$\begin{aligned}
\frac{d\,x_1}{\sqrt{P(x_1)}}+\frac{d\,x_2}{\sqrt{P(x_2)}} &= -i\frac{\sqrt{(s_1-k)(s_1+k)}}{s_1-s_2}dt\\
\frac{d\,x_1}{\sqrt{P(x_1)}}-\frac{d\,x_2}{\sqrt{P(x_2)}} &= -i\frac{\sqrt{(s_2-k)(s_2+k)}}{s_1-s_2}dt.
\end{aligned}$$

From discriminant separability, one gets (see Corollary 1 from [15]):

$$\begin{aligned}\frac{dx_1}{\sqrt{P(x_1)}} + \frac{dx_2}{\sqrt{P(x_2)}} &= \frac{ds_1}{\sqrt{J(s_1)}}\\ -\frac{dx_1}{\sqrt{P(x_1)}} + \frac{dx_2}{\sqrt{P(x_2)}} &= \frac{ds_2}{\sqrt{J(s_2)}}\end{aligned} \tag{19}$$

and finally

$$\begin{aligned}\frac{ds_1}{\sqrt{\Phi(s_1)}} + \frac{ds_2}{\sqrt{\Phi(s_2)}} &= 0\\ \frac{s_1\,ds_1}{\sqrt{\Phi(s_1)}} + \frac{s_2\,ds_2}{\sqrt{\Phi(s_2)}} &= \imath dt,\end{aligned} \tag{20}$$

where

$$\Phi(s) = J(s)(s-k)(s+k),$$

where Φ is a polynomial of degree up to six.

Thus, relations (20) define the Abel map on a genus 2 curve

$$y^2 = \Phi(s).$$

□

2.2 An example of a new integrable systems of Kowalevski type

Now, we are going to present a new example of systems of the Kowalevski type.

Let us consider the next system of differential equations:

$$\begin{aligned}\dot{p} &= -rq\\ \dot{q} &= -rp - \gamma_3\\ \dot{r} &= -2q(2p+1) - 2\gamma_2\\ \dot{\gamma}_1 &= 2(q\gamma_3 - r\gamma_2)\\ \dot{\gamma}_2 &= 2(p\gamma_3 - r\gamma_1)\\ \dot{\gamma}_3 &= 2(p^2-q^2)q - 2q\gamma_1 + 2p\gamma_2.\end{aligned} \tag{21}$$

After a change of variables

$$\begin{aligned}x_1 &= p+q,\\ x_2 &= p-q,\\ e_1 &= x_1^2 + \gamma_1 + \gamma_2,\end{aligned}$$

$$e_2 = x_2^2 + \gamma_1 - \gamma_2,$$

system (21) becomes

$$\begin{aligned}
\dot{x}_1 &= -rx_1 - \gamma_3 \\
\dot{x}_2 &= rx_2 + \gamma_3 \\
\dot{e}_1 &= -2re_1 \\
\dot{e}_2 &= 2re_2 \\
\dot{r} &= -x_1 + x_2 - e_1 + e_2 \\
\dot{\gamma}_3 &= x_2 e_1 - x_1 e_2.
\end{aligned} \tag{22}$$

We can write down the first integrals of system (22) in the next form

$$\begin{aligned}
r^2 &= 2(x_1 + x_2) + e_1 + e_2 + h \\
r\gamma_3 &= -x_1 x_2 - x_2 e_1 - x_1 e_2 - \frac{g_2}{4} \\
\gamma_3^2 &= x_2^2 e_1 + x_1^2 e_2 - \frac{g_3}{2} \\
e_1 \cdot e_2 &= k^2.
\end{aligned} \tag{23}$$

Then, like in Kowalevski's case, from integrals (23) we get a relation in the form of (17)

$$\begin{aligned}
&(x_1 - x_2)^2 e_1 e_2 + \left(2x_1^3 + hx_1^2 - \frac{g_2}{2}x_1 - \frac{g_3}{2}\right) e_2 + \left(2x_2^3 + hx_2^2 - \frac{g_2}{2}x_2 - \frac{g_3}{2}\right) e_1 \\
&- \left(x_1^2 x_2^2 + x_1 x_2 \frac{g_2}{2} + g_3(x_1 + x_2 + \frac{h}{2}) + \frac{g_2^2}{16}\right) = 0.
\end{aligned} \tag{24}$$

Suppose here that constant of motion $h = 0$ just to illustrate integration procedure directly in terms of Weierstrass $\wp$ function. Following the procedure described in Theorem 1 we get

$$\begin{aligned}
\frac{dx_1}{\sqrt{P(x_1)}} + \frac{dx_2}{\sqrt{P(x_2)}} &= \frac{ds_1}{\sqrt{P(s_1)}} \\
\frac{dx_1}{\sqrt{P(x_1)}} - \frac{dx_2}{\sqrt{P(x_2)}} &= \frac{ds_2}{\sqrt{P(s_2)}}
\end{aligned} \tag{25}$$

where $P(x)$ denotes the polynomial

$$P(x) = 2x^3 - \frac{g_2}{2}x - \frac{g_3}{2}, \tag{26}$$

and s_1, s_2 are the solutions of quadratic equation in s:

$$\begin{aligned}
\mathcal{F}(x_1, x_2, s) &:= A(x_1, x_2)s^2 + B(x_1, x_2)s + C(x_1, x_2) \\
&= (x_1 - x_2)^2 s^2 + \left(-2x_1 x_2(x_1 + x_2) + \frac{g_2}{2}(x_1 + x_2) + g_3\right) s \\
&+ x_1^2 x_2^2 + x_1 x_2 \frac{g_2}{2} + g_3(x_1 + x_2) + \frac{g_2^2}{16} = 0.
\end{aligned} \tag{27}$$

Without assumption that constant $h = 0$, in system of equations (25) polynomial would be $P(x) = 2x^3 + hx^2 - \frac{g_2}{2}x - \frac{g_3}{2}$. With simple linear change of variables

$$x_i \mapsto x_i - \frac{h}{6}, \quad s \mapsto s - \frac{h}{6}$$

equations for $h \neq 0$ transform into (25).

Finally, we get

Proposition 1 *The system of differential equations defined by (21) is integrated through the solutions of the system*

$$\begin{aligned} \frac{ds_1}{\sqrt{\Phi(s_1)}} + \frac{ds_2}{\sqrt{\Phi(s_2)}} &= 0 \\ \frac{s_1\,ds_1}{\sqrt{\Phi(s_1)}} + \frac{s_2\,ds_2}{\sqrt{\Phi(s_2)}} &= 2\,dt, \end{aligned} \tag{28}$$

where

$$\Phi(s) = P(s)(s-k)(s+k).$$

2.3 Explicit integration in genus two theta functions

This Section is devoted to explicit integration of the system (21). Integration procedure will be done in two ways. The first one is based on Kowalevski [24] and uses properties of elliptic functions. The second one follows Kötter [23] and Golubev [18]. A generalization of Kötter transformation was derived in [15] for a polynomial $P(x)$ of degree four. Here we will reformulate such a transformation for $P(x)$ of degree three.

We are going to consider here, as in [24], the case where the zeros l_i, $i = 1, 2, 3$ of the polynomial P of degree three are real and $l_1 > l_2 > l_3$. Denote

$$l = (l_1 - l_2)(l_2 - l_3)(l_3 - l_1).$$

Following Kowalevski, we consider functions

$$P_i = \sqrt{(s_1 - l_i)(s_2 - l_i)}, \quad i = 1, 2, 3 \tag{29}$$

and

$$P_{ij} = P_i P_j \left(\frac{\dot{s}_1}{(s_1 - l_i)(s_1 - l_j)} + \frac{\dot{s}_2}{(s_2 - l_i)(s_2 - l_j)} \right). \tag{30}$$

Then by simple calculations one gets

$$
\begin{aligned}
\dot{P}_{ij} &= \frac{1}{2} P_i P_j \\
\dot{P}_1 &= \frac{P_3 P_{13} - P_2 P_{12}}{2(l_2 - l_3)} \\
\dot{P}_2 &= \frac{P_1 P_{12} - P_3 P_{23}}{2(l_3 - l_1)} \\
\dot{P}_3 &= \frac{P_2 P_{23} - P_1 P_{13}}{2(l_1 - l_2)}.
\end{aligned}
\tag{31}
$$

We will now derive expressions for p, q, r, γ_1, γ_2, γ_3 in terms of P_i, P_{ij} functions for i, $j = 1$, 2, 3.

Denote by

$$
du_i = \frac{dx_i}{\sqrt{4x_i^3 - g_2 x_i - g_3}}, \quad i = 1, 2.
$$

Then

$$
x_i = \wp(u_i),
$$

and dividing both equations of system (25) with $\sqrt{2}$ we see that

$$
s_1 = \wp(u_1 + u_2), \quad s_2 = \wp(u_1 - u_2).
$$

We use the next properties of $\wp$-function, see [24]:

$$
\begin{aligned}
\wp(u_1) + \wp(u_2) &= -2 \frac{(l_2^2 - l_3^2)P_1 + (l_3^2 - l_1^2)P_2 + (l_1^2 - l_2^2)P_3}{(l_2 - l_3)P_1 + (l_3 - l_1)P_2 + (l_1 - l_2)P_3}, \\
\wp(u_1) - \wp(u_2) &= \frac{-2l}{(l_2 - l_3)P_1 + (l_3 - l_1)P_2 + (l_1 - l_2)P_3}, \\
\wp(u_1) \cdot \wp(u_2) &= -\Big[\frac{(l_2 - l_3)(l_1^2 + l_2 l_3)P_1 + (l_3 - l_1)(l_2^2 + l_1 l_3)P_2}{(l_2 - l_3)P_1 + (l_3 - l_1)P_2 + (l_1 - l_2)P_3} \\
&\quad + \frac{(l_1 - l_2)(l_3^2 + l_1 l_2)P_3}{(l_2 - l_3)P_1 + (l_3 - l_1)P_2 + (l_1 - l_2)P_3} \Big].
\end{aligned}
\tag{32}
$$

After some calculations, we get the next expressions for variables p, q, r, γ_1, γ_2, γ_3 in terms of P_i and P_{ij} functions for $i, j = 1, 2, 3$:

$$
p = \frac{x_1 + x_2}{2} = \frac{\wp(u_1) + \wp(u_2)}{2} = -\frac{(l_2^2 - l_3^2)P_1 + (l_3^2 - l_1^2)P_2 + (l_1^2 - l_2^2)P_3}{(l_2 - l_3)P_1 + (l_3 - l_1)P_2 + (l_1 - l_2)P_3},
\tag{33}
$$

$$
q = \frac{x_1 - x_2}{2} = \frac{\wp(u_1) - \wp(u_2)}{2} = -\frac{l}{(l_2 - l_3)P_1 + (l_3 - l_1)P_2 + (l_1 - l_2)P_3},
\tag{34}
$$

$$
r = -\frac{\dot{p}}{q} = \frac{1}{2} \frac{(l_1 - l_2)P_{12} + (l_2 - l_3)P_{23} + (l_3 - l_1)P_{13}}{(l_2 - l_3)P_1 + (l_3 - l_1)P_2 + (l_1 - l_2)P_3},
\tag{35}
$$

$$
\begin{aligned}
\gamma_1 &= \frac{r^2}{2} - p^2 - q^2 - 2p \\
&= \frac{\left((l_1-l_2)P_{12} + (l_2-l_3)P_{23} + (l_3-l_1)P_{13}\right)^2}{8\left((l_2-l_3)P_1 + (l_3-l_1)P_2 + (l_1-l_2)P_3\right)^2} \\
&- \frac{\left((l_2^2-l_3^2)P_1 + (l_3^2-l_1^2)P_2 + (l_1^2-l_2^2)P_3\right)^2 + l^2}{\left((l_2-l_3)P_1 + (l_3-l_1)P_2 + (l_1-l_2)P_3\right)^2} \\
&+ 2\frac{(l_2^2-l_3^2)P_1 + (l_3^2-l_1^2)P_2 + (l_1^2-l_2^2)P_3}{(l_2-l_3)P_1 + (l_3-l_1)P_2 + (l_1-l_2)P_3},
\end{aligned} \tag{36}
$$

$$
\begin{aligned}
\gamma_2 &= -q(2p+1) - \frac{\dot{r}}{2} \\
&= \frac{l}{\left((l_2-l_3)P_1 + (l_3-l_1)P_2 + (l_1-l_2)P_3\right)^2} \\
&\cdot \big[(l_2 - l_3 - 2l_2^2 + 2l_3^2)P_1 + (l_3 - l_1 - 2l_3^2 + 2l_1^2)P_2 \\
&+ (l_1 - l_2 - 2l_1^2 + 2l_2^2)P_3\big] \\
&- \frac{\left((l_2-l_3)P_2P_3 + (l_3-l_1)P_1P_3 + (l_1-l_2)P_1P_2\right)}{8(l_2-l_3)P_1 + (l_3-l_1)P_2 + (l_1-l_2)P_3} \\
&- \frac{(l_2-l_3)P_{23} + (l_3-l_1)P_{13} + (l_1-l_2)P_{12}}{8\left((l_2-l_3)P_1 + (l_3-l_1)P_2 + (l_1-l_2)P_3\right)^2} \\
&\cdot \left(P_3P_{13} + P_1P_{12} + P_2P_{23} - P_2P_{12} - P_1P_{13} - P_3P_{23}\right),
\end{aligned} \tag{37}
$$

$$
\begin{aligned}
\gamma_3 &= -(\dot{q} + rp) \\
&= \frac{1}{2}\frac{(l_2^2-l_3^2)P_1 + (l_3^2-l_1^2)P_2 + (l_1^2-l_2^2)P_3}{\left((l_2-l_3)P_1 + (l_3-l_1)P_2 + (l_1-l_2)P_3\right)^2} \\
&\cdot \left((l_1-l_2)P_{12} + (l_2-l_3)P_{23} + (l_3-l_1)P_{13}\right) \\
&- \frac{l}{2}\frac{P_3P_{13} + P_1P_{12} + P_2P_{23} - P_2P_{12} - P_1P_{13} - P_3P_{23}}{\left((l_2-l_3)P_1 + (l_3-l_1)P_2 + (l_1-l_2)P_3\right)^2}.
\end{aligned} \tag{38}
$$

The expressions in terms of theta functions for P_i and P_{ij} for $i, j = 1, 2, 3$ are given in the Appendix.

Now, we will perform integration following Kötter [23] and Golubev [18]. First, we will formulate an extension of Kötter's transformation for a degree three polynomial $P(x) = 2x^3 - \frac{g_2}{2}x - \frac{g_3}{2}$.

Proposition 2 *For a polynomial $\mathcal{F}(x_1, x_2, s)$ given with formula (27), there exist polynomials $\alpha(x_1, x_2, s)$, $\beta(x_1, x_2, s)$, $P(s)$ such that the following identity*

$$
\mathcal{F}(x_1, x_2, s) = \alpha^2(x_1, x_2, s) + P(s)\beta(x_1, x_2, s), \tag{39}
$$

is satisfied. The polynomials are defined by the formulae:

$$\begin{aligned}
\alpha(x_1, x_2, s) &= 2s^2 + s(x_1 + x_2) - x_1 x_2 - \frac{g_2}{4} \\
\beta(x_1, x_2, s) &= -2(x_1 + x_2 + s) \\
P(s) &= 2s^3 - \frac{g_2}{2}s - \frac{g_3}{2},
\end{aligned}$$

where P coincides with the polynomial from formula (26).

Proof. The proof follows by a direct calculation. $\square$

Define

$$\hat{\mathcal{F}}(s) = \frac{\mathcal{F}(x_1, x_2, s)}{(x_1 - x_2)^2},$$

and consider the identity

$$\hat{\mathcal{F}}(s) = (s-u)^2 + (s-u)\hat{\mathcal{F}}'(u) + \hat{\mathcal{F}}(u).$$

Then, from (39) we get

$$\begin{aligned}
&(s-u)^2(x_1 - x_2)^2 + 2(s-u)\left(u(x_1 - x_2)^2 + \frac{B(x_1, x_2)}{2}\right) \\
&+ \alpha^2(x_1, x_2, u) + P(u)\beta(x_1, x_2, u) = 0.
\end{aligned}$$

Corollary 1 *(a) The solutions s_1, s_2 of the last equation in s satisfy an identity in u:*

$$(s_1 - u)(s_2 - u) = \frac{\alpha^2(x_1, x_2, u)}{(x_1 - x_2)^2} + P(u)\frac{\beta(x_1, x_2, u)}{(x_1 - x_2)^2},$$

where $P(u)$ is a polynomial defined with (26).

(b) Functions P_i satisfy

$$P_i = \frac{\alpha(x_1, x_2, l_i)}{x_1 - x_2} = \left(2l_i^2 - \frac{g_2}{4}\right)\frac{1}{x_1 - x_2} + l_i\frac{x_1 + x_2}{x_1 - x_2} - \frac{x_1 x_2}{x_1 - x_2}. \tag{40}$$

Now we introduce a more convenient notation

$$\begin{aligned}
X &= \frac{x_1 x_2}{x_1 - x_2}, \\
Y &= \frac{1}{x_1 - x_2}, \\
Z &= \frac{x_1 + x_2}{x_1 - x_2}.
\end{aligned}$$

Lemma 1 *The quantities X, Y, Z satisfy the system of linear equations*

$$\begin{aligned}
-X + \left(2l_1^2 - \frac{g_2}{4}\right) Y + l_1 Z &= P_1 \\
-X + \left(2l_2^2 - \frac{g_2}{4}\right) Y + l_2 Z &= P_2 \\
-X + \left(2l_3^2 - \frac{g_2}{4}\right) Y + l_3 Z &= P_3.
\end{aligned} \tag{41}$$

The solutions of the system (41) are

$$\begin{aligned}
Y &= \frac{(l_2 - l_3)P_1 + (l_3 - l_1)P_2 + (l_1 - l_2)P_3}{-2l}, \\
Z &= \frac{(l_2^2 - l_3^2)P_1 + (l_3^2 - l_1^2)P_2 + (l_1^2 - l_2^2)P_3}{l}, \\
X &= -\Big(\frac{(g_2 + 8l_2 l_3)(l_2 - l_3)P_1 + (g_2 + 8l_3 l_1)(l_3 - l_1)P_2}{8l} \\
&\quad + \frac{(g_2 + 8l_1 l_2)(l_1 - l_2)P_3}{8l}\Big).
\end{aligned} \tag{42}$$

Using Viète formulae for polynomial $P(x)$ we can rewrite X in the form

$$X = \frac{(l_2 - l_3)(l_1^2 + l_2 l_3)P_1 + (l_3 - l_1)(l_2^2 + l_1 l_3)P_2 + (l_1 - l_2)(l_3^2 + l_1 l_2)P_3}{2l}.$$

Now, from the expressions for X, Y, Z we get

$$\begin{aligned}
q &= \frac{x_1 - x_2}{2} = \frac{1}{2Y}, \\
p &= \frac{x_1 + x_2}{2} = \frac{Z}{2Y}.
\end{aligned}$$

Expressions for r and γ_i, $i = 1, 2, 3$ can be derived in terms of P_i, P_{ij} functions from equations of the system (21), see formulae (33)-(38).

2.4 Further analysis of systems of Kowalevski-type. First integrals

Now, we perform further analysis of possible first integrals of systems of Kowalevski type. We are looking for a system in variables $x_1, x_2, e_1, e_2, r, \gamma_3$ that satisfies relations of the form

$$\begin{aligned}
r^2 &= E + p_2 e_1 + p_1 e_2 \\
r\gamma_3 &= F - q_2 e_1 - q_1 e_2 \\
\gamma_3^2 &= G + r_2 e_1 + r_1 e_2 \\
e_1 \cdot e_2 &= k^2.
\end{aligned} \tag{43}$$

The next Lemma shows that the assumptions defining systems of the Kowalevski type (15, 16, 17), in a sense, determine the particular integrals.

Lemma 2 *A system which reduces to (15), (16), (17) with $f_i = rx_i + \gamma_3$, next relations are satisfied:*

$$
\begin{aligned}
r^2 &= \frac{P(x_1) + P(x_2) \pm 2\sqrt{P(x_1)P(x_2) + AC} + Ae_1 + Ae_2}{(x_1 - x_2)^2} \\
r\gamma_3 &= -\frac{x_2P(x_1) + x_1P(x_2) \pm (x_1 + x_2)\sqrt{P(x_1)P(x_2) + AC}}{(x_1 - x_2)^2} \\
&\quad - \frac{x_2Ae_1 + x_1Ae_2}{(x_1 - x_2)^2} \\
\gamma_3^2 &= \frac{x_2^2P(x_1) + x_1^2P(x_2) \pm 2x_1x_2\sqrt{P(x_1)P(x_2) + AC} + x_2^2Ae_1 + x_1^2Ae_2}{(x_1 - x_2)^2} \\
e_1 \cdot e_2 &= k^2.
\end{aligned}
\tag{44}
$$

Proof. Replacing (43) into condition (16) with $f_i = rx_i + \gamma_3$, we get

$$r^2x_i^2 + 2x_ir\gamma_3 + \gamma_3^2 = P(x_i) + e_iA(x_1, x_2).$$

Collecting coefficients with e_i, we obtain a system with unknowns $p_1, p_2, q_1, q_2, r_1, r_2, E, F, G$:

$$
\begin{aligned}
p_2x_1^2 - 2q_2x_1 + r_2 &= A(x_1, x_2) \\
p_1x_1^2 - 2q_1x_1 + r_1 &= 0 \\
Ex_1^2 + 2Fx_1 + G &= P(x_1) \\
p_2x_2^2 - 2q_2x_2 + r_2 &= 0 \\
p_1x_2^2 - 2q_1x_2 + r_1 &= A(x_1, x_2) \\
Ex_2^2 + 2Fx_2 + G &= P(x_2).
\end{aligned}
\tag{45}
$$

The solutions of the system are:

$$
\begin{aligned}
r_1 &= \frac{x_1(x_1A + 2q_1x_2(x_1 - x_2))}{x_1^2 - x_2^2}, \\
r_2 &= -\frac{x_2(x_2A + 2q_2x_1(x_2 - x_1))}{x_1^2 - x_2^2}, \\
p_1 &= -\frac{A + 2q_1(x_2 - x_1)}{x_1^2 - x_2^2}, \\
p_2 &= \frac{A + 2q_2(x_1 - x_2)}{x_1^2 - x_2^2}, \\
F &= \frac{E(x_2^2 - x_1^2) + P(x_1) - P(x_2)}{x_1 - x_2}, \\
G &= \frac{Ex_1x_2(x_1 - x_2) + x_1P(x_2) - x_2P(x_1)}{x_1 - x_2}.
\end{aligned}
\tag{46}
$$

The second assumption is that the relation

$$(E + p_2 e_1 + p_1 e_2)(G + r_2 e_1 + r_1 e_2) - (F - q_2 e_1 - q_1 e_2)^2 = 0 \tag{47}$$

can be rewritten in the form (17). According to (17), the coefficients of e_i^2 should vanish, so we obtain:

$$q_1 = \frac{x_1 A}{(x_1 - x_2)^2}, \quad q_2 = \frac{x_1 A}{(x_1 - x_2)^2}. \tag{48}$$

Replacing these results into (47) it becomes

$$\frac{A}{(x_1 - x_2)^2} \big[A e_1 e_2 + P(x_2) e_1 + P(x_1) e_2 \\ - \frac{(x_1 - x_2)^2}{A} \big(\frac{(x_1 - x_2)^2 E^2}{4} - \frac{(P(x_1) + P(x_2))E}{2} + \frac{(P(x_1) - P(x_2))^2}{4(x_1 - x_2)^2} \big) \big] = 0.$$

Finally, solving

$$\frac{(x_1 - x_2)^2}{A} \left[\frac{(x_1 - x_2)^2 E^2}{4} - \frac{(P(x_1) + P(x_2))E}{2} + \frac{(P(x_1) - P(x_2))^2}{4(x_1 - x_2)^2} \right] = C,$$

we get

$$E = \frac{P(x_1) + P(x_2) \pm 2\sqrt{P(x_1)P(x_2) + AC}}{(x_1 - x_2)^2}. \tag{49}$$

Then we can write (47) in the form of (17):

$$\frac{A(x_1, x_2)(A(x_1, x_2) e_1 e_2 + e_1 P(x_2) + e_2 P(x_1) - C(x_1, x_2))}{(x_1 - x_2)^2} = 0.$$

□

Corollary 2 *In the case when $A(x_1, x_2)$, , $B(x_1, x_2)$, $C(x_1, x_2)$ are the coefficients of a discriminantly separable polynomial (14), the relations (44) can be written in the next form:*

$$\begin{aligned}
r^2 &= \frac{P(x_1) + P(x_2) \pm B(x_1, x_2) + A(x_1, x_2) e_1 + A(x_1, x_2) e_2}{(x_1 - x_2)^2} \\
r\gamma_3 &= -\frac{x_2 P(x_1) + x_1 P(x_2) \pm (x_1 + x_2)\frac{B(x_1, x_2)}{2}}{(x_1 - x_2)^2} \\
&\quad - \frac{x_2 A(x_1, x_2) e_1 + x_1 A(x_1, x_2) e_2}{(x_1 - x_2)^2} \\
\gamma_3^2 &= \frac{x_2^2 P(x_1) + x_1^2 P(x_2) \pm x_1 x_2 B(x_1, x_2) + x_2^2 A(x_1, x_2) e_1 + x_1^2 A(x_1, x_2) e_2}{(x_1 - x_2)^2} \\
e_1 \cdot e_2 &= k^2.
\end{aligned} \tag{50}$$

Different choices of signs in relations (44) determine different systems. Thus, systems of such type appear in pairs.

Corollary 3 *In order the relations (44) be the first integrals for system from Lemma 2 in the form of polynomial, $A(x_1, x_2)$ must be of the form const $\cdot (x_1 - x_2)^2$.*

In the same way as in Lemma 2 we can obtain possible first integrals for systems with functions $f_i = x_i^{m_i} \cdot r + x_i^{n_i} \cdot \gamma_3$, $i = 1, 2$ and $m_i, n_i \in \mathbb{Z}$.

Proposition 3 *For a system which reduces to (15), (16), (17) with*

$$f_i = x_i^{m_i} \cdot r + x_i^{n_i} \cdot \gamma_3,\ i = 1, 2,$$

$m_i, n_i \in \mathbb{Z}$ and at least one of conditions $m_1 \neq n_1$, $m_2 \neq n_2$ is valid, then following relations in the form of (43) are satisfied for next expressions:

$$\begin{aligned}
p_1 &= \frac{A(x_1,x_2)x_1^{2n_1}}{(x_1^{m_1}x_2^{n_2} - x_2^{m_2}x_1^{n_1})^2}, \quad p_2 = \frac{A(x_1,x_2)x_2^{2n_2}}{(x_1^{m_1}x_2^{n_2} - x_2^{m_2}x_1^{n_1})^2},\\
q_1 &= \frac{A(x_1,x_2)x_1^{n_1+m_1}}{(x_1^{m_1}x_2^{n_2} - x_2^{m_2}x_1^{n_1})^2}, \quad q_2 = \frac{A(x_1,x_2)x_1^{n_2+m_2}}{(x_1^{m_1}x_2^{n_2} - x_2^{m_2}x_1^{n_1})^2},\\
r_1 &= \frac{A(x_1,x_2)x_1^{2m_1}}{(x_1^{m_1}x_2^{n_2} - x_2^{m_2}x_1^{n_1})^2}, \quad r_2 = \frac{A(x_1,x_2)x_2^{2m_2}}{(x_1^{m_1}x_2^{n_2} - x_2^{m_2}x_1^{n_1})^2},\\
E_i &= \frac{x_2^{2n_2}P(x_1) + x_1^{2n_1}P(x_2) \pm B(x_1,x_2)x_1^{n_1}x_2^{n_2}}{(x_1^{m_1}x_2^{n_2} - x_2^{m_2}x_1^{n_1})^2},\ i = 1, 2\\
F_i &= \frac{E_i(x_1^{2m_1}x_2^{2n_2} - x_1^{2n_1}x_2^{2m_2}) + x_1^{2n_1}P(x_2) - x_2^{2n_2}P(x_1)}{2x_1^{n_1}x_2^{n_2}(x_1^{n_1}x_2^{m_2} - x_1^{m_1}x_2^{n_2})},\ i = 1, 2\\
G_i &= \frac{E_i x_1^{m_1}x_2^{m_2}(x_1^{m_1}x_2^{n_2} - x_1^{n_1}x_2^{m_2}) + x_1^{m_1+n_1}P(x_2) - x_2^{m_2+n_2}P(x_1)}{x_1^{n_1}x_2^{n_2}(x_1^{m_1}x_2^{n_2} - x_1^{n_1}x_2^{m_2})},\ i = 1, 2.
\end{aligned}$$

Here $B(x_1, x_2)$ denotes a function in two variables such that

$$B^2(x_1, x_2) = 4A(x_1, x_2)C(x_1, x_2) + 4P(x_1)P(x_2).$$

Example 1 *Now we will apply Lemma 2 to the Kowalevski case, which has been briefly presented in the Introduction. Denote by*

$$A = (x_1 - x_2)^2,$$

$$B = x_1^2x_2^2 - 6l_1x_1x_2 - 2cl(x_1 + x_2) - (c^2 - k^2),$$

$$C = 6l_1x_1^2x_2^2 + (c^2 - k^2)(x_1 + x_2)^2 + 4lcx_1x_2(x_1 + x_2) - 6l_1(c^2 - k^2) + 4l^2c^2,$$

$$P(x) = x^4 - 6l_1x^2 - 4clx_2 - c^2 + k^2$$

the polynomials that appear in the Kowalevski fundamental equation (11). Applying Lemma 2 we get

$$p_1 = 1,\, p_2 = 1$$

$$q_1 = x_1,\, q_2 = x_2$$

$$r_1 = x_1^2,\, r_2 = x_2^2.$$

The expressions for E, F, G *we will denote by* E_i, F_i, G_i, $i = 1, 2$ *depending on choice of a sign in the expression for* E*:*

$$\begin{aligned} E_1 &= 6l_1 - (x_1 + x_2)^2 \\ F_1 &= x_1x_2(x_1 + x_2) + 2cl \\ G_1 &= -x_1^2x_2^2 + c^2 - k^2, \end{aligned}$$

and

$$\begin{aligned} E_2 &= -\frac{x_1^4 - 6x_1^2l_1 - 8clx_1 + x_2^4 - 6l_1x_2^2 - 8clx_2 - 4c^2 + 4k^2 + 2x_2^2x_1^2 - 12l_1x_1x_2}{(x_1 - x_2)^2} \\ F_2 &= \frac{1}{(x_1 - x_2)^2}(x_1^4x_2 + x_1^3x_2^2 - 12x_1^2l_1x_2 - 2x_1^2cl + 2x_1(k^2 - c^2) - 12lcx_1x_2 \\ &\quad + x_1x_2^4 + x_1^2x_2^3 - 12x_1l_1x_2^2 - 2lcx_2^2 + 2x_2(k^2 - c^2)) \\ G_2 &= -\frac{x_1^2x_2^2(x_1^2 + x_2^2) + 2x_2^3x_1^3 - 24l_1x_1^2x_2^2 + (k^2 - c^2)(x_1 + x_2)^2}{(x_1 - x_2)^2} \\ &\quad + \frac{8lcx_1x_2(x_1 + x_2)}{(x_1 - x_2)^2}. \end{aligned}$$

The expressions E_1, F_1, G_1 *correspond to the original Kowalevski case.*

The expressions E_2, F_2, G_2 *correspond to a new system with constant parameters* c, l, l_1*. By differentiating the first and the third relation (43) for* E_2, F_2, G_2 *instead of* E, F, G*, for a system which reduces to (15), (16), (17) with* $f_i = rx_i + \gamma_3$ *we get*

$$\begin{aligned} \dot{r} &= \frac{\imath}{2r(x_1 - x_2)^3}\big[(x_2^5 + x_1^5)r - 8(e_1e_2 - c^2)\gamma_3 + 24lcrx_1x_2 + 12l_1\gamma_3(x_1 + x_2)^2 \\ &\quad - 2x_1^2x_2^2r(x_1 + x_2) - 3x_1x_2r(x_1^3 + x_2^3) - 4x_1x_2\gamma_3(x_1^2 + x_2^2) \\ &\quad - 4(e_1e_2 - c^2)r(x_1 + x_2) + 4lcr(x_1^2 + x_2^2) + 24l_1x_1x_2r(x_1 + x_2) \\ &\quad + 16cl\gamma_3(x_1 + x_2) + m(x_1 - x_2)^3(e_1 + e_2)\imath\big], \end{aligned}$$

$$\begin{aligned} \dot{\gamma}_3 &= \frac{-\imath}{2\gamma_3(x_1 - x_2)^3}\big[x_1x_2\gamma_3(x_1^4 + x_2^4) + 4x_1^3x_2^3r(x_1 + x_2) + 2(e_1e_2 - c^2)\gamma_3(x_1^2 + x_2^2) \\ &\quad + 2x_1^2x_2^2\gamma_3(x_1^2 + x_2^2 + x_1x_2) - 16lc\gamma_3x_1x_2(x_1 + x_2) + 4(e_1e_2 - c^2)x_1x_2(x_1 + x_2) \\ &\quad - 32clx_1^2x_2^2r - 24l_1x_1x_2\gamma_3(x_1^2 + x_2^2) - 24l_1x_1^2x_2^2r(x_1 + x_2) + 4e_1e_2x_1x_2\gamma_3 \\ &\quad + (mx_1\imath + \gamma_3 + x_1r)x_1(x_1 - x_2)^3e_2 - (mx_2\imath + \gamma_3 + x_2r)x_2(x_1 - x_2)^3e_1\big]. \end{aligned}$$

Finally, by differentiating of the second relation

$$r\gamma_3 = F_2 - x_2e_1 - x_1e_2$$

and by substituting obtained expressions for $\dot{r}$ and $\dot{\gamma}_3$, we get value for so far unknown function m:

$$\begin{aligned}
m = \imath r + &\frac{1}{(x_1-x_2)^3(rx_1+\gamma_3)^2e_2 - (x_1-x_2)^3(rx_2+\gamma_3)^2e_1}\\
&[4x_1x_2r^3(x_1^2x_2^2(x_1+x_2) + (e_1e_2-c^2)(x_1+x_2) - 6l_1x_1x_2(x_1+x_2) - 8clx_1x_2)\\
&+ 4\gamma_3r^2((e_1e_2-c^2)(x_1^2+x_2^2+4x_1x_2) + 2x_1^2x_2^2(x_1^2+x_2^2+x_1x_2)\\
&- 9l_1x_1x_2(x_1+x_2)^2 - 12lcx_1x_2(x_1+x_2)) + 4\gamma_3^2r(x_1x_2(x_1^3+x_2^3)\\
&+ 2x_1^2x_2^2(x_1+x_2) - 14clx_1x_2 - 15l_1x_1x_2(x_1+x_2) - 3l_1(x_1^3+x_2^3)\\
&+ 3(e_1e_@-c^2)(x_1+x_2) - 5lc(x_1^2+x_2^2)) + 4\gamma_3^3(x_1x_2(x_1^2+x2^2) - 3l_1(x_1^2+x_2^2)\\
&- 4cl(x_1+x_2) - 6l_1x_1x_2 + 2(e_1e_2-c^2))].
\end{aligned}$$

From the previous example and Lemma 2 we see that for a given discriminantly separable polynomial $\mathcal{F}(x_1, x_2, s)$, there are always two integrable systems that satisfy (15), (16), (17) corresponding to two choices of functions E, F, G.

2.5 Another method for obtaining systems of Kowalevski-type

Now, we will show another method for obtaining new systems that reduce to (15), (16), (17), with possible first integrals in the form (43).

We search for functions $E, F, G, p_i, q_i, r_i,\ i = 1, 2$ such that the relation following from

$$(E + p_2e_1 + p_1e_2)(G + r_2e_1 + r_1e_2) - (F - q_2e_1 - q_1e_2)^2 = 0$$

leads to a relation of the form (17). Thus, one set of conditions follows from annulation of the coefficients with e_1^2, e_2^2. This gives us the first conditions:

$$p_1r_1 = q_1^2,\ \ p_2r_2 = q_2^2. \tag{51}$$

We come to the relation

$$\tilde{P}_2e_1 + \tilde{P}_1e_2 = C(x_1, x_2) - e_1e_2A(x_1, x_2) \tag{52}$$

where

$$\begin{aligned} A &= p_1r_2 + p_2r_1 - 2q_1q_2 \\ C &= F^2 - EG \end{aligned} \tag{53}$$

and

$$\begin{aligned}\tilde{P}_1 &= r_1E + 2q_1F + p_1G \\ \tilde{P}_2 &= r_2E + 2q_2F + p_2G.\end{aligned} \tag{54}$$

Now, let us assume the following:

$$B_1 = E\sqrt{r_1r_2} + F(\sqrt{p_1r_2} + \sqrt{p_2r_1}) + G\sqrt{p_1p_2}. \tag{55}$$

Then, we have the following

Lemma 3 *The functions $A, B_1, C, \tilde{P}_1, \tilde{P}_2$ defined above, satisfy the identity:*

$$B_1^2 - AC = \tilde{P}_1\tilde{P}_2.$$

Lemma 4 *For the system with relations in the form (43) satisfied with (51)-(54), functions f_i defined by*

$$f_i = \sqrt{r_i}r + \sqrt{p_i}\gamma_3, \quad i = 1,\, 2$$

also satisfy assumption (16).

Proof. Proof is done by a straightforward calculation.

$$\begin{aligned} f_1^2 &= r_1r^2 + 2\sqrt{p_1r_1}r\gamma_3 + p_1\gamma_3^2 \\ &= r_1(E + p_2e_1 + p_1e_2) + 2q_1(F - q_2e_1 - q_1e_2) + p_1(G + r_2e_1 + r_1e_2) \\ &= Er_1 + Gp_1 + 2q_1F + e_1(r_1p_2 + p_1r_2 - 2q_1q_2) \\ &+ e_2(r_1p_1 + p_1r_1 - 2q_1^2) \\ &= \tilde{P}_1 + e_1(r_1p_2 + p_1r_2 - 2\sqrt{r_1p_2r_2p_1}) \\ &= \tilde{P}_1 + e_1A. \end{aligned} \tag{56}$$

In the same way, we get

$$f_2^2 = \tilde{P}_2 + e_2A.$$

□

Now, let us introduce the second assumption:

$$\begin{aligned} r_i &= x_i^2, \quad i = 1, 2 \\ p_i &= (x_i - a)^2, \quad i = 1, 2 \\ q_i &= x_i(x_i - a), \quad i = 1, 2 \end{aligned} \tag{57}$$

The last, crucial condition, in order to get a relation of the form (17) from (52), is

$$\tilde{P}_1 = P(x_1), \;\; \tilde{P}_2 = P(x_2),$$

which means that the functions $\tilde{P}_1$ and $\tilde{P}_2$ are equal to a polynomial P of one variable, x_1 in the former and x_2 in the later case.

To satisfy the last requirement, we need to guess the right form of the functions E, F, G. Let us, finally, assume:

$$\begin{aligned} E &= -2(x_1-a)^2(x_2-a)^2 + C_1 \\ F &= x_1(x_1-a)(x_2-a)^2 + x_2(x_2-a)(x_1-a)^2 + C_2 \\ G &= -2x_1x_2(x_2-a)(x_1-a) + C_3. \end{aligned} \tag{58}$$

Theorem 2 *Polynomials E, F, G, p_i, q_i, r_i define a completely integrable system, with*

$$f_i = x_i r + (x_i - a)\gamma_3, \quad i = 1, 2. \tag{59}$$

The system is explicitly integrated on the pinched genus two curve in theta-functions of a generalized Jacobian of an elliptic curve.

Similar systems appeared in a slightly different context in the works of Appel'rot, Mlodzeevskii, Delone in their study of degenerations of the Kowalevski top (see [3], [26],[13]). In particular, we may construct *Delone-type* solutions of the last system:

$$s_1 = 0, \quad s_2 = \wp\left(\frac{i}{4}(t-t_0)\right).$$

Proposition 4 *The equations of motion for a completely integrable system described in Theorem 2, with functions f_i, $i = 1, 2$ defined by (59) are*

$$\begin{aligned} \dot{x}_1 &= -\frac{\imath}{2} f_1 \\ \dot{x}_2 &= \frac{\imath}{2} f_2 \\ \dot{e}_1 &= -m e_1 \\ \dot{e}_2 &= m e_2 \\ \dot{r} &= (x_2-x_1)(x_2-a)(x_1-a)a\imath + \frac{\imath f_2(x_2-a)}{2r} e_1 - \frac{\imath f_1(x_1-a)}{2r} e_2 \\ &\quad - \frac{e_1(x_2-a)^2 - e_2(x_1-a)^2}{2r} m \\ \dot{\gamma}_3 &= \frac{e_2 x_1^2 - e_1 x_2^2}{2\gamma_3} m - \frac{\imath f_2 x_2 e_1}{2\gamma_3} - \frac{\imath f_1 x_1 e_2}{2\gamma_3} \\ &\quad - \frac{\imath(\gamma_3(a-x_1)(a-x_2) - r x_1 x_2)(x_2-x_1)a}{2\gamma_3}, \end{aligned}$$

with

$$m = \imath \frac{-(r+\gamma_3) f_1^2 e_2 - (\gamma_3 a(r+\gamma_3) - x_2(\gamma_3^2 + 2r\gamma_3 - r^2)) f_2 e_1 - a r (x_1 - x_2) f_1 f_2}{f_2^2 e_1 - f_1^2 e_2}. \tag{60}$$

Proof. System of equations described in Theorem 2 is

$$\begin{aligned}
\dot{x}_1 &= -\frac{\imath}{2}(x_1 r + (x_1 - a)\gamma_3) \\
\dot{x}_2 &= \frac{\imath}{2}(x_2 r + (x_2 - a)\gamma_3) \\
\dot{e}_1 &= -m e_1 \\
\dot{e}_2 &= m e_2,
\end{aligned}$$

with first integrals

$$r^2 = -2(x_1 - a)^2(x_2 - a)^2 + C_1 + (x_2 - a)^2 e_1 + (x_1 - a)^2 e_2, \tag{61}$$

$$r\gamma_3 = x_1(x_1-a)(x_2-a)^2 + x_2(x_2-a)(x_1-a)^2 + C_2 - x_2(x_2-a)e_1 - x_1(x_1-a)e_2, \tag{62}$$

$$\gamma_3^2 = -2x_1x_2(x_2 - a)(x_1 - a) + C_3 + x_2^2 e_1 + x_1^2 e_2, \tag{63}$$

$$e_1 e_2 = k^2.$$

Differentiating first integrals (61) and (63) we get equations for $\dot{r}$ and $\dot{\gamma}_3$. Then, differentiating integral (62) we get

$$\begin{aligned}
r\dot{\gamma}_3 + \dot{r}\gamma_3 &= \dot{x}_1(x_1 - a)(x_2 - a)^2 + x_1\dot{x}_1(x_2 - a)^2 + x_1(x_1 - a)2(x_2 - a)\dot{x}_2 \\
&+ \dot{x}_2(x_2 - a)(x_1 - a)^2 + x_2\dot{x}_2(x_1 - a)^2 + x_2(x_2 - a)2(x_1 - a)\dot{x}_1 - x_1\dot{x}_1 e_2 \\
&- \dot{x}_1(x_1 - a)e_2 - m x_1(x_1 - a)e_2 - \dot{x}_2(x_2 - a)e_1 - x_2\dot{x}_2 e_1 + m x_2(x_2 - a)e_1.
\end{aligned}$$

Substituting obtained values for $\dot{r}$ and $\dot{\gamma}_3$ we get equation for m with solution (60). □

Proposition 5 *The system of differential equations defined by Proposition 4 is integrated to through the solutions of the system*

$$\begin{aligned}
\frac{ds_1}{s_1\sqrt{\Phi_1(s_1)}} + \frac{ds_2}{s_2\sqrt{\Phi_1(s_2)}} &= 0 \\
\frac{ds_1}{\sqrt{\Phi_1(s_1)}} + \frac{ds_2}{\sqrt{\Phi_1(s_2)}} &= \frac{i}{2}dt,
\end{aligned} \tag{64}$$

where

$$\Phi_1(s) = s(s - e_4)(s - e_5)$$

is the polynomial of degree 3.

By choosing expressions for p_i, r_i, $i = 1, 2$ and then finding E, F, G such that $\tilde{P}_i = Er_i + 2Fq_i + Gp_i$ be polynomial, we can obtain new examples of systems of Kowalevski type. We will give a few more examples after classification of strongly discriminantly separable polynomials, in Section 6.

3 A deformation of the Kowalevski top

In this Section we are going to derive explicit solutions in genus two theta-functions of Jurdjevic elasticae [20] and similar systems [21], [22]. First, we show that we can get the elasticae from the Kowalevski top by using the simplest gauge transformations.

Consider a discriminantly separable polynomial

$$\mathcal{F}(x_1, x_2, s) := s^2 A + sB + C$$

where

$$\begin{aligned} &A = (x_1 - x_2)^2, \\ &B = -2(Ex_1x_2 + F(x_1 + x_2) + G), \\ &C = F^2 - EG. \end{aligned} \tag{65}$$

A simple affine gauge transformation

$$s \mapsto t + \alpha$$

transforms $\mathcal{F}(x_1, x_2, s)$ into

$$\mathcal{F}_\alpha(x_1, x_2, t) = t^2 A_\alpha + tB_\alpha + C_\alpha,$$

with

$$\begin{aligned} A_\alpha &= A \\ B_\alpha &= B + 2\alpha A \\ C_\alpha &= C + \alpha B + \alpha^2 A. \end{aligned} \tag{66}$$

Next, we denote

$$\begin{aligned} F_\alpha &= F + \alpha F_1 \\ E_\alpha &= E + \alpha E_1 \\ G_\alpha &= G + \alpha G_1. \end{aligned} \tag{67}$$

From

$$C_\alpha = F_\alpha^2 - E_\alpha G_\alpha,$$

by equating powers of α, we get

$$\begin{aligned} B &= 2FF_1 - E_1G - EG_1 \\ A &= F_1^2 - E_1G_1. \end{aligned} \tag{68}$$

From (65) one obtains

$$\begin{aligned} F_1 &= -(x_1 + x_2) \\ G_1 &= 2x_1x_2 \\ E_1 &= 2. \end{aligned} \tag{69}$$

One easily checks

$$F_1^2 - E_1 G_1 = A,$$

$$\begin{aligned}
E_\alpha &= 6l_1 - (x_1 + x_2)^2 + 2\alpha \\
F_\alpha &= 2cl + x_1 x_2 (x_1 + x_2) - \alpha(x_1 + x_2) \\
G_\alpha &= c^2 - k^2 - x_1^2 x_2^2 + 2\alpha x_1 x_2.
\end{aligned} \tag{70}$$

Jurdjevic considered a deformation of the Kowalevski case associated to a Kirchhoff elastic problem, see [20]. The systems are defined by the Hamiltonians

$$H = M_1^2 + M_2^2 + 2M_3^2 + \gamma_1$$

where the deformed Poisson structures $\{\cdot,\cdot\}_\tau$ are defined by

$$\{M_i, M_j\}_\tau = \epsilon_{ijk} M_k, \quad \{M_i, \gamma_j\}_\tau = \epsilon_{ijk}\gamma_k, \quad \{\gamma_i, \gamma_j\}_\tau = \tau\epsilon_{ijk} M_k,$$

where the deformation parameter takes values $\tau = 0, 1, -1$. These structures correspond to $e(3)$, $so(4)$ and $so(3,1)$ respectively. The classical Kowalevski case corresponds to the case $\tau = 0$. The systems with $\tau = -1, 1$ have been considered by St. Petersburg's school (Komarov, Kuznetsov [22]) in 1990's, and they have been rediscovered by several authors in meantime. Here, we are giving explicit formulae in theta-functions for solutions of these systems.

Denote

$$\begin{aligned}
e_1 &= x_1^2 - (\gamma_1 + i\gamma_2) + \tau \\
e_2 &= x_2^2 - (\gamma_1 - i\gamma_2) + \tau,
\end{aligned}$$

where

$$x_{1,2} = \frac{M_1 \pm iM_2}{2}.$$

The integrals of motion

$$\begin{aligned}
I_1 &= e_1 e_2 \\
I_2 &= H \\
I_3 &= \gamma_1 M_1 + \gamma_2 M_2 + \gamma_3 M_3 \\
I_4 &= \gamma_1^2 + \gamma_2^2 + \gamma_3^2 + \tau(M_1^2 + M_2^2 + M_3^2)
\end{aligned}$$

may be rewritten in the form (43)

$$\begin{aligned}
k^2 &= I_1 = e_1 \cdot e_2 \\
M_3^2 &= e_1 + e_2 + \hat{E}(x_1, x_2) \\
M_3\gamma_3 &= -x_2 e_1 - x_1 e_2 + \hat{F}(x_1, x_2) \\
\gamma_3^2 &= x_2^2 e_1 + x_1^2 e_2 + \hat{G}(x_1, x_2),
\end{aligned}$$

where

$$\begin{aligned}
\hat{G}(x_1, x_2) &= -x_1^2 x_2^2 - 2\tau x_1 x_2 - 2\tau(I_1 - \tau) + \tau^2 - I_2 \\
\hat{F}(x_1, x_2) &= (x_1 x_2 + \tau)(x_1 + x_2) + I_3 \\
\hat{E}(x_1, x_2) &= -(x_1 + x_2)^2 + 2(I_1 - \tau).
\end{aligned}$$

Proposition 6 *An affine gauge transformation*

$$s \mapsto t + \alpha$$

transforms the Kowalevski top to Jurdjevic elasticae according to the formulae

$$\begin{aligned}\tau &= -\alpha\\ I_1 &= 3l_1\\ I_3 &= 2cl\\ I_2 &= c^2 - k^2 + 2\alpha(3l_1+\alpha) + \alpha^2.\end{aligned}$$

We can apply generalized Kötter transformation derived in [15] to obtain expressions for M_i, γ_i in terms of P_i and P_{ij} functions for $i, j = 1, 2, 3$. First we will write down the equations of motion for Jurdjevic elasticae:

$$\begin{aligned}\dot{M}_1 &= 2M_2M_3\\ \dot{M}_2 &= -2M_1M_3 + \gamma_3\\ \dot{M}_3 &= -\gamma_2\\ \dot{\gamma}_1 &= -2M_2\gamma_3 + 4M_3\gamma_2\\ \dot{\gamma}_2 &= 2M_1\gamma_2 - 4M_3\gamma_1 + \tau M_3\\ \dot{\gamma}_3 &= -2M_1\gamma_2 + 2M_2\gamma_1 - \tau M_2.\end{aligned} \tag{71}$$

Now we introduce the following notation:

$$\begin{aligned}R(x_1,x_2) &= \hat{E}x_1x_2 + \hat{F}(x_1+x_2) + \hat{G},\\ R_1(x_1,x_2) &= \hat{E}\hat{G} - \hat{F}^2,\\ P(x_i) &= \hat{E}x_i^2 + 2\hat{F}x_i + \hat{G}, \quad i = 1,\, 2.\end{aligned}$$

Lemma 5 *For polynomial $\mathcal{F}(x_1,x_2,s)$ given with*

$$\mathcal{F}(x_1,x_2,s) = (x_1-x_2)^2s^2 - 2R(x_1,x_2)s - R_1(x_1,x_2),$$

there exist polynomials $A(x_1,x_2,s)$, $B(x_1,x_2,s)$, $f(s)$, $A_0(s)$ such that the following identity

$$\mathcal{F}(x_1,x_2,s)A_0(s) = A^2(x_1,x_2,s) + f(s)B(x_1,x_2,s), \tag{72}$$

is satisfied. The polynomials are defined by the formulae:

$$\begin{aligned}A_0(s) &= 2s + 2I_1 - 2\tau\\ f(s) &= 2s^3 + 2(I_1-3\tau)s^2 + (8\tau^2 - 4\tau(I_1-\tau) - 2I_2 - 4\tau I_1)s\\ &\quad + (I_1-\tau)(6\tau^2 - 2I_2 - 4\tau I_1) - I_3^2 + 2(I_1-\tau)\tau^2\\ A(x_1,x_2,s) &= A_0(s)(x_1x_2 - s) + I_3(x_1+x_2) + 2\tau(I_1-\tau) + 2\tau s\\ B(x_1,x_2,s) &= (x_1+x_2)^2 - 2s - 2I_1 + 2\tau.\end{aligned}$$

Denote by m_i the zeros of polynomial f and

$$P_i = \sqrt{(s_1 - m_i)(s_2 - m_i)} \;\; i = 1,\, 2,\, 3.$$

The same way as in Corollary 1 we get

$$P_i = \frac{\sqrt{A_0(m_i)}(x_1x_2 - m_i)}{x_1 - x_2} + \frac{I_3(x_1 + x_2) + 2\tau(I_1 - \tau + m_i)}{(x_1 - x_2)\sqrt{A_0(m_i)}}, \quad i = 1,\, 2,\, 3. \tag{73}$$

Now we introduce a more convenient notation

$$\begin{aligned}
X &= \frac{x_1x_2}{x_1 - x_2}\\
Y &= \frac{1}{x_1 - x_2}\\
Z &= \frac{I_3(x_1 + x_2) + 2\tau(I_1 - \tau)}{x_1 - x_2}\\
n_i &= A_0(m_i) = 2m_i + 2I_1 - 2\tau, \quad i = 1,\, 2,\, 3.
\end{aligned}$$

Relations (73) can be rewritten as a system of linear equations

$$\begin{aligned}
X + Ym_1\left(\frac{2\tau}{n_1} - 1\right) + \frac{Z}{n_1} &= \frac{P_1}{\sqrt{n_1}}\\
X + Ym_2\left(\frac{2\tau}{n_2} - 1\right) + \frac{Z}{n_2} &= \frac{P_2}{\sqrt{n_2}}\\
X + Ym_3\left(\frac{2\tau}{n_3} - 1\right) + \frac{Z}{n_3} &= \frac{P_3}{\sqrt{n_3}}.
\end{aligned}$$

The solutions of the previous system are

$$\begin{aligned}
Y &= -\sum_{i=1}^{3}\frac{\sqrt{n_i}P_i}{f'(m_i)}\\
X &= -\sum_{i=1}^{3}\frac{P_i\sqrt{n_i}}{f'(m_i)}\left(m_j + m_k + I_1 - 2\tau\right)\\
Z &= \sum_{i=1}^{3}\frac{2\sqrt{n_i}P_i}{f'(m_i)}\left(\frac{n_j \cdot n_k}{4} + \tau(\tau - I_1)\right),
\end{aligned} \tag{74}$$

with (i, j, k) - a cyclic permutation of $(1, 2, 3)$.

Finally, we obtain

Proposition 7 *The solution of the system of differential equations (71) in*

terms of P_i, P_{ij} functions is given with

$$
\begin{aligned}
M_1 &= -\frac{\sum_{i=1}^3 \frac{2\sqrt{n_i}P_i}{f'(m_i)}(\frac{n_j \cdot n_k}{4}+\tau(\tau-I_1))}{I_3\sum_{i=1}^3 \frac{\sqrt{n_i}P_i}{f'(m_i)}} - \frac{2\tau(I_1-\tau)}{I_3} \\
M_2 &= -\frac{1}{\imath\sum_{i=1}^3 \frac{\sqrt{n_i}P_i}{f'(m_i)}} \\
M_3 &= \frac{\sum_{k=1}^3 \frac{n_k\sqrt{n_i n_j}P_{ij}}{f'(m_k)}}{2\imath(\sum_{i=1}^3 \frac{\sqrt{n_i}P_i}{f'(m_i)})} \\
\gamma_1 &= I_2 + \frac{1}{2}\left(\frac{\sum_{k=1}^3 \frac{n_k\sqrt{n_i n_j}P_{ij}}{f'(m_k)}}{\sum_{i=1}^3 \frac{\sqrt{n_i}P_i}{f'(m_i)}}\right)^2 - \frac{4\sum_{i=1}^3 \frac{P_i\sqrt{n_i}}{f'(m_i)}(m_j+m_k+I_1-2\tau)}{\sum_{i=1}^3 \frac{P_i\sqrt{n_i}}{f'(m_i)}} \\
\gamma_2 &= -\frac{(\sum_{k=1}^3 \frac{n_k\sqrt{n_i n_j}}{f'(m_k)}\frac{P_iP_j}{2})\cdot(\sum_{i=1}^3 \frac{\sqrt{n_i}P_i}{f'(m_i)})}{2\imath\left(\sum_{i=1}^3 \frac{\sqrt{n_i}P_i}{f'(m_i)}\right)^2} \\
&\quad + \frac{(\sum_{k=1}^3 \frac{n_k\sqrt{n_i n_j}P_{ij}}{f'(m_k)})\cdot(\sum_{i=1}^3 \frac{\sqrt{n_i}}{f'(m_i)}\frac{P_kP_{ik}-P_jP_{ij}}{2(m_j-m_k)})}{2\imath\left(\sum_{i=1}^3 \frac{\sqrt{n_i}P_i}{f'(m_i)}\right)^2} \\
\gamma_3 &= \frac{\sum_{k=1}^3 \frac{\sqrt{n_i n_j}P_{ij}}{f'(m_k)}}{2\imath\sum_{i=1}^3 \frac{\sqrt{n_i}P_i}{f'(m_i)}}.
\end{aligned}
$$

Formulae expressing P_i, P_{ij} in terms of theta-functions are given in the Appendix. This gives explicit formulae for the elasticae.

4 Another system of Kowalevski type

In this section we will consider another class of systems of Kowalevski type. We consider a situation analogue to that from the beginning of the Section 2.1. The only difference is that the systems we are going to consider now, reduce to (15), where

$$
\begin{aligned}
f_1^2 &= P(x_1) - \frac{C}{e_2} \\
f_2^2 &= P(x_2) - \frac{C}{e_1}.
\end{aligned}
\tag{75}
$$

The next Theorem is an analogue of Theorem 1. Thus, the new class of systems also has a striking property of jumping genus in integration procedure.

Theorem 3 *Given a system which reduces to (15), where*

$$
\begin{aligned}
f_1^2 &= P(x_1) - \frac{C}{e_2} \\
f_2^2 &= P(x_2) - \frac{C}{e_1}
\end{aligned}
\tag{76}
$$

and integrals reduce to (17); A, C, P form a discriminantly separable polynomial $\mathcal{F}$ given with (14). Then the system is linearized on the Jacobian of the curve

$$y^2 = J(z)(z-k)(z+k),$$

where J is a polynomial factor of the discriminant of $\mathcal{F}$ as a polynomial in x_1 and k is a constant such that

$$e_1 e_2 = k^2.$$

Proof. Again, from the equations of motion on e_i we get

$$e_1 e_2 = k^2,$$

with some constant k. Now, we get

$$\left(\sqrt{e_1}\sqrt{P(x_2)} \pm \sqrt{e_2}\sqrt{P(x_1)}\right)^2 = C(x_1, x_2) - k^2 A(x_1, x_2) \pm 2\sqrt{P(x_1)P(x_2)}k.$$

Dividing with $A(x_1, x_2) \neq 0$, we get

$$\left(\sqrt{e_1}\sqrt{\frac{P(x_2)}{A}} \pm \sqrt{e_2}\sqrt{\frac{P(x_1)}{A}}\right)^2 = \frac{C}{A} - k^2 \pm 2\frac{\sqrt{P(x_1)P(x_2)}}{A}k$$

From the last equation, we obtain

$$
\begin{aligned}
\left(\sqrt{e_1}\sqrt{\frac{P(x_2)}{A}} + \sqrt{e_2}\sqrt{\frac{P(x_1)}{A}}\right)^2 &= (s_1 - k)(s_2 + k) \\
\left(\sqrt{e_1}\sqrt{\frac{P(x_2)}{A}} - \sqrt{e_2}\sqrt{\frac{P(x_1)}{A}}\right)^2 &= (s_1 + k)(s_2 - k)
\end{aligned}
$$

where s_1, s_2 are the solutions of the quadratic equation

$$\mathcal{F}(x_1, x_2, s) = 0$$

in s. From the last equations, dividing with $k = \sqrt{e_1 e_2}$ we get

$$
\begin{aligned}
2\sqrt{\frac{P(x_2)}{e_2 A}} &= \frac{1}{k}\left(\sqrt{(s_1+k)(s_2-k)} + \sqrt{(s_1-k)(s_2+k)}\right) \\
2\sqrt{\frac{P(x_1)}{e_1 A}} &= \frac{1}{k}\left(\sqrt{(s_1+k)(s_2-k)} - \sqrt{(s_1-k)(s_2+k)}\right).
\end{aligned}
$$

Since s_i are solutions of the quadratic equation $F(x_1, x_2, s_i) = 0$, we get

$$\begin{aligned} s_1 + s_2 &= -\frac{B}{A} \\ s_2 - s_1 &= \frac{2\sqrt{P(x_1)P(x_2)}}{A} \\ s_1 \cdot s_2 &= \frac{C}{A}. \end{aligned} \tag{77}$$

From the last equations, we get

$$(s_1 - s_2)^2 = 4\frac{P(x_1)P(x_2)}{A^2}.$$

Using the last equation, we have

$$\begin{aligned} f_1^2 &= P(x_1) - \frac{C(x_1, x_2)}{e_2} \\ &= \frac{(s_1 - s_2)^2 A^2}{4P(x_2)} - \frac{C}{e_2} \\ &= \frac{A^2}{4P(x_2)}\left[(s_1 - s_2)^2 - \frac{C}{A}\frac{4P(x_2)}{e_2 A}\right] \\ &= \frac{P(x_1)}{(s_1 - s_2)^2}\left[(s_1 - s_2)^2 - s_1 s_2 \frac{1}{k^2}\left(\sqrt{(s_1 + k)(s_2 - k)} + \sqrt{(s_1 - k)(s_2 + k)}\right)^2\right] \\ &= \frac{P(x_1)}{(s_1 - s_2)^2}\left[s_1^2 - 2s_1 s_2 + s_2^2 - \frac{2s_1 s_2}{k^2}\left(s_1 s_2 - k^2 + \sqrt{(s_1^2 - k^2)(s_2^2 - k^2)}\right)\right] \\ &= \frac{P(x_1)}{k^2(s_1 - s_2)^2}\left[k^2(s_1^2 + s_2^2) - 2s_1^2 s_2^2 - 2s_1 s_2\sqrt{(s_1^2 - k^2)(s_2^2 - k^2)}\right] \\ &= -\frac{P(x_1)}{k^2(s_1 - s_2)^2}\left[s_2\sqrt{s_1^2 - k^2} + s_1\sqrt{s_2^2 - k^2}\right]^2. \end{aligned}$$

Similarly

$$f_2^2 = -\frac{P(x_2)}{k^2(s_1 - s_2)^2}\left[s_2\sqrt{s_1^2 - k^2} - s_1\sqrt{s_2^2 - k^2}\right]^2.$$

From the last two equations and from the equations of motion, we get

$$\begin{aligned} \frac{d\,x_1}{\sqrt{P(x_1)}} + \frac{d\,x_2}{\sqrt{P(x_2)}} &= \frac{-s_1\sqrt{s_2^2 - k^2}}{k(s_1 - s_2)}dt \\ \frac{d\,x_1}{\sqrt{P(x_1)}} - \frac{d\,x_2}{\sqrt{P(x_2)}} &= \frac{-s_2\sqrt{s_1^2 - k^2}}{k(s_1 - s_2)}dt. \end{aligned}$$

Discriminant separability condition (see Corollary 1 from [15]) gives

$$\begin{aligned} \frac{dx_1}{\sqrt{P(x_1)}} - \frac{dx_2}{\sqrt{P(x_2)}} &= -\frac{ds_1}{\sqrt{J(s_1)}} \\ \frac{dx_1}{\sqrt{P(x_1)}} + \frac{dx_2}{\sqrt{P(x_2)}} &= \frac{ds_2}{\sqrt{J(s_2)}}. \end{aligned} \tag{78}$$

Finally

$$\begin{aligned}\frac{ds_1}{\sqrt{\Phi(s_1)}}+\frac{ds_2}{\sqrt{\Phi(s_2)}}&=-\frac{d\,t}{k}\\ \frac{s_1\,ds_1}{\sqrt{\Phi(s_1)}}+\frac{s_2\,ds_2}{\sqrt{\Phi(s_2)}}&=0,\end{aligned}\tag{79}$$

where

$$\Phi(s)=J(s)(s-k)(s+k),$$

is a polynomial of degree up to six. □

5 Strongly discriminantly separable polynomials, classification. Quad-graphs

5.1 Classification of strongly discriminantly separable polynomials of degree two in three variables

In this section we will classify strongly discriminantly separable polynomials $\mathcal{F}(x_1,x_2,x_3)\in\mathbb{C}[x_1,x_2,x_3]$ which are of degree two in each variable, modulo a gauge group of the following fractional-linear transformations

$$\begin{aligned}x_1&\mapsto\frac{ax_1+b}{cx_1+d}\\ x_2&\mapsto\frac{ax_2+b}{cx_2+d}\\ x_3&\mapsto\frac{ax_3+b}{cx_3+d}.\end{aligned}\tag{80}$$

Let

$$\mathcal{F}(x_1,x_2,x_3)=\sum_{i,j,k=0}^{2}a_{ijk}x_1^ix_2^jx_3^k\tag{81}$$

be a strongly discriminantly separable polynomial with

$$\mathcal{D}_{x_i}\mathcal{F}(x_j,x_k)=P(x_j)P(x_k),\quad (i,j,k)=c.p.(1,2,3).\tag{82}$$

Here, by $\mathcal{D}_{x_i}\mathcal{F}(x_j,x_k)$ we denote the discriminant of $\mathcal{F}$ considered as a quadratic polynomial in x_i.

By replacing (81) into (82) for a given polynomial

$$P(x)=Ax^4+Bx^3+Cx^2+Dx+E,$$

we get a system of 75 equations of degree two with 27 unknowns a_{ijk}.

Theorem 4 *Strongly discriminantly separable polynomials $\mathcal{F}(x_1, x_2, x_3)$ satisfying (82) modulo fractional linear transformations are exhausted by the following list depending on distribution of roots of a non-zero polynomial $P(x)$:*

(A) *four simple zeros, for canonical form $P(x) = (k^2x^2 - 1)(x^2 - 1)$,*

$$\mathcal{F}_A = \frac{1}{2}(-k^2x_1^2 - k^2x_2^2 + 1 + k^2x_1^2x_2^2)x_3^2 + (1-k^2)x_1x_2x_3 + \frac{1}{2}(x_1^2 + x_2^2 - k^2x_1^2x_2^2 - 1),$$

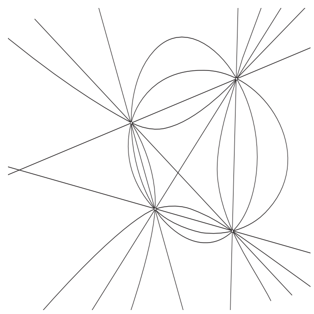

Figure 1: Pencil with four simple points

(B) *two simple zeros and one double zero, for canonical form $P(x) = x^2 - e^2$, $e \neq 0$,*

$$\mathcal{F}_B = x_1x_2x_3 + \frac{e}{2}(x_1^2 + x_2^2 + x_3^2 - e^2),$$

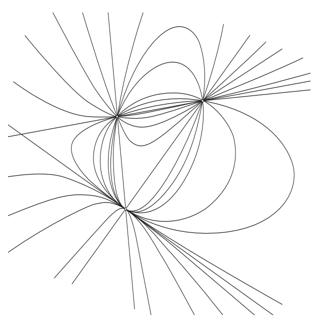

Figure 2: Pencil with one double and two simple points

(C) *two double zeros, for canonical form $P(x) = x^2$,*

$$\mathcal{F}_{C1} = \lambda x_1^2x_3^2 + \mu x_1x_2x_3 + \nu x_2^2, \quad \mu^2 - 4\lambda\nu = 1,$$
$$\mathcal{F}_{C2} = \lambda x_1^2x_2^2x_3^2 + \mu x_1x_2x_3 + \nu, \quad \mu^2 - 4\lambda\nu = 1,$$

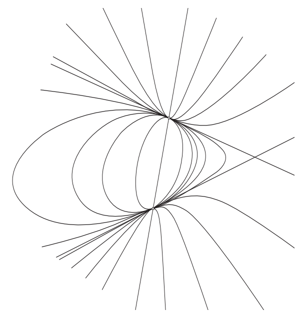

Figure 3: Pencil with two double points

(D) *one simple and one triple zero, for canonical form $P(x) = x$,*

$$\mathcal{F}_D = -\frac{1}{2}(x_1x_2 + x_2x_3 + x_1x_3) + \frac{1}{4}(x_1^2 + x_2^2 + x_3^2),$$

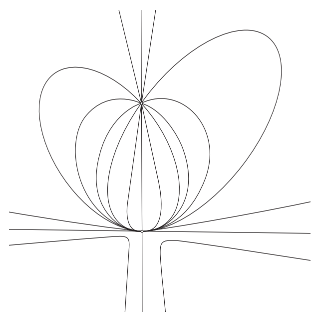

Figure 4: Pencil with one simple and one triple point

(E) *one quadruple zero, for canonical form $P(x) = 1$,*

$$\begin{aligned}
\mathcal{F}_{E1} &= \lambda(x_1 + x_2 + x_3)^2 + \mu(x_1 + x_2 + x_3) + \nu, \quad \mu^2 - 4\lambda\nu = 1,\\
\mathcal{F}_{E2} &= \lambda(x_2 + x_3 - x_1)^2 + \mu(x_2 + x_3 - x_1) + \nu, \quad \mu^2 - 4\lambda\nu = 1,\\
\mathcal{F}_{E3} &= \lambda(x_1 + x_3 - x_2)^2 + \mu(x_1 + x_3 - x_2) + \nu, \quad \mu^2 - 4\lambda\nu = 1,\\
\mathcal{F}_{E4} &= \lambda(x_1 + x_2 - x_3)^2 + \mu(x_1 + x_2 - x_3) + \nu, \quad \mu^2 - 4\lambda\nu = 1.
\end{aligned}$$

Corresponding pencils of conics are of types (1,1,1,1), (1,1,2), (2,2), (1,3) and (4) in cases (A)-(E) respectively. (These pencils are presented on Figures 1-5 respectively.)

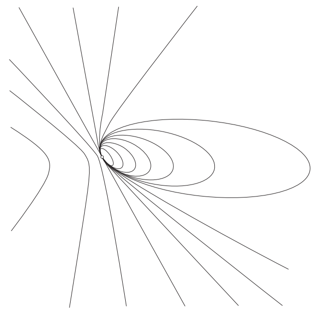

Figure 5: Pencil with one quadruple point

Proof. The system we ought to solve is obtained by equating coefficients with $x_i^{\alpha_i} x_j^{\alpha_j}; i,j = 1,2,3; \alpha_i, \alpha_j = 0,1,2,3,4$ in equations:

$$\left(\sum_{i,j=0}^{2} a_{1ij} x_2^i x_3^j\right)^2 - 4\left(\sum_{i,j=0}^{2} a_{2ij} x_2^i x_3^j\right)\left(\sum_{i,j=0}^{2} a_{0ij} x_2^i x_3^j\right) = P(x_2)P(x_3),$$

$$\left(\sum_{i,j=0}^{2} a_{i1j} x_1^i x_3^j\right)^2 - 4\left(\sum_{i,j=0}^{2} a_{i2j} x_1^i x_3^j\right)\left(\sum_{i,j=0}^{2} a_{i0j} x_1^i x_3^j\right) = P(x_1)P(x_3),$$

$$\left(\sum_{i,j=0}^{2} a_{ij1} x_1^i x_2^j\right)^2 - 4\left(\sum_{i,j=0}^{2} a_{ij2} x_1^i x_2^j\right)\left(\sum_{i,j=0}^{2} a_{ij0} x_1^i x_2^j\right) = P(x_1)P(x_2).$$

By a straightforward calculation, for example in the case

$$P(x) = (k^2x^2 - 1)(x^2 - 1),$$

we get polynomials:

$$\begin{aligned}
\mathcal{F}_{A1_{1,2}} = \pm\big((&-\frac{k^2}{2}x_1^2 - \frac{k^2}{2}x_2^2 + \frac{1}{2} + \frac{k^2}{2}x_1^2x_2^2)x_3^2 + (1-k^2)x_1x_2x_3 \\
&+ \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2 - \frac{k^2}{2}x_1^2x_2^2 - \frac{1}{2}\big), \\
\mathcal{F}_{A2_{1,2}} = \pm\big((&-\frac{k^2}{2}x_1^2 - \frac{k^2}{2}x_2^2 + \frac{1}{2} + \frac{k^2}{2}x_1^2x_2^2)x_3^2 + (-1+k^2)x_1x_2x_3 \\
&+ \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2 - \frac{k^2}{2}x_1^2x_2^2 - \frac{1}{2}\big),
\end{aligned}$$

$$\mathcal{F}_{A3_{1,2}} = \pm\big((\frac{k}{2} - \frac{k}{2}x_1^2 - \frac{k}{2}x_2^2 + \frac{k^3}{2}x_1^2x_2^2)x_3^2 + (1-k^2)x_1x_2x_3$$
$$+ \frac{k}{2}x_1^2 + \frac{k}{2}x_2^2 - +\frac{k}{2}x_1^2x_2^2 - +\frac{1}{2k})\big),$$
$$\mathcal{F}_{A4_{1,2}} = \pm\big((\frac{k}{2} - \frac{k}{2}x_1^2 - \frac{k}{2}x_2^2 + \frac{k^3}{2}x_1^2x_2^2)x_3^2 + (-1+k^2)x_1x_2x_3$$
$$+ \frac{k}{2}x_1^2 + \frac{k}{2}x_2^2 - +\frac{k}{2}x_1^2x_2^2 - +\frac{1}{2k})\big).$$

It is clear that polynomials $\mathcal{F}_{A1_{1,2}}$ and $\mathcal{F}_{A2_{1,2}}$ are fractionally-linearly equivalent with a common transformation $x_i = -X_i, i = 1, 2, 3$. The same case is for $\mathcal{F}_{A3_{1,2}}$ and $\mathcal{F}_{A4_{1,2}}$.

Next, with transformations

$$x_i = \frac{1}{kX_i}, i = 1, 2, 3,$$

the polynomial $\mathcal{F}_{A3_1}(x_1, x_2, x_3)$ turns into

$$\frac{\mathcal{F}_{A1_2}(X_1, X_2, X_3)}{k^3X_1^2X_2^2X_3^2}.$$

Thus, all obtained polynomials are gauge equivalent to $\mathcal{F}_A$.

Other polynomials of the list (B)-(E) are obtained in the same manner. $\square$

Remark 1 *Notice that polynomials $\mathcal{F}_{E2}, \mathcal{F}_{E3}, \mathcal{F}_{E4}$ are related to $\mathcal{F}_{E1}$ with transformations $x_1 \mapsto -x_1$ for $\mathcal{F}_{E2}$, $x_2 \mapsto -x_2$ for $\mathcal{F}_{E3}$, $x_3 \mapsto -x_3$ for $\mathcal{F}_{E4}$.*

Proposition 8 *When $P(x) \equiv 0$ all strongly discriminantly separable polynomials $\mathcal{F}(x_1, x_2, x_3)$ are equivalent with*

$$\mathcal{F}_0(x_1, x_2, x_3) = (\delta_1x_1x_2x_3 + \delta_2(x_1x_2 + x_1x_3 + x_2x_3) + \delta_3(x_1 + x_2 + x_3) + \delta_4)^2$$

modulo possibly nonequal fractionally-linearly transformations

$$x_i \mapsto \frac{a_ix_i + b_i}{c_ix_i + d_i}, \quad i = 1, 2, 3.$$

We consider more closely the case when $P(x)$ has four simple zeros. The transformation

$$x_i = -\frac{12k^2y_i + k^2 - 5}{12k^2y_i - 5k^2 + 1}, \quad i = 1, 2, 3 \tag{83}$$

brings polynomial $\mathcal{F}_A(x_1, x_2, x_3)$ into the form

$$\frac{54(k-1)^3(k+1)^3}{(12k^2y_1 - 5k^2 + 1)^2(12k^2y_2 - 5k^2 + 1)^2(12k^2y_2 - 5k^2 + 1)^2}\hat{\mathcal{F}}_A(y_1, y_2, y_3),$$

with

$$\hat{\mathcal{F}}_A(y_1,y_2,y_3) = 6912k^8\big((y_1y_2+y_1y_3+y_2y_3+\frac{14k^2+k^4+1}{48k^4})^2$$
$$-(y_1+y_2+y_3)(4y_1y_2y_3-\frac{(k^2+1)(k^2-6k+1)(k^2+6k+1)}{216k^6})\big)$$
$$= -6912k^8(y_1+y_2+y_3)(4y_1y_2y_3-g_3)-\big(y_1y_2+y_1y_3+y_2y_3+\frac{g_2}{4}\big)^2.$$

Here g_2 and g_3 are well known coefficients of the Weierstrass normal form $4x^3 - g_2x - g_3$. For a polynomial $P(x) = A + 4Bx + 6Cx^2 + 4Dx^3 + Ex^4$, they are given with $g_2 = AE - 4BD + 3C^2$, $g_3 = ACE + 2BDC - AD^2 - B^2E - C^3$.

Proposition 9 *In the case when polynomial $P(x)$ has four simple zeros all strongly discriminantly separable polynomials $\mathcal{F}(x_1,x_2,x_3)$ satisfying (82) are modulo fractionally-linearly transformations equivalent to*

$$\mathcal{F}(x_1,x_2,x_3) = (x_1+x_2+x_3)(4x_1x_2x_3-g_3)-\Big(x_1x_2+x_1x_3+x_2x_3+\frac{g_2}{4}\Big)^2. \tag{84}$$

Notice here that when we expand (84) as polynomial of x_3 it becomes

$$\begin{aligned}\mathcal{F}(x_1,x_2,x_3) = -(x_1-x_2)^2x_3^2+(2x_1^2x_2+2x_1x_2^2-g_3-x_1\frac{g_2}{2}-x_2\frac{g_2}{2})x_3 \\ -\frac{g_2^2}{16}-x_1g_3-x_1^2x_2^2-x_2g_3-\frac{g_2}{2}x_1x_2.\end{aligned} \tag{85}$$

Comparing with [8] we get

Corollary 4 *If a polynomial P has four simple zeros, all strongly discriminantly separable polynomials $\mathcal{F}$ satisfying (82) are equivalent to two-valued groups*

$$(x_1x_2+x_2x_3+x_1x_3)(4+g_3x_1x_2x_3) = \Big(x_1+x_2+x_3-\frac{g_2x_1x_2x_3}{4}\Big)^2.$$

5.2 From discriminantly separable polynomials to integrable quad-graphs

Now, we will establish connections between discriminantly separable polynomials and the theory of integrable systems on quad graphs from works of Adler, Bobenko, Suris [1],[2]. Moreover, we are going to construct an integrable quad graph associated to an arbitrary given discriminantly separable polynomial.

Recall that the basic building blocks of systems on quad-graphs are the equations on quadrilaterals of the form

$$Q(x_1, x_2, x_3, x_4) = 0 \tag{86}$$

where Q is a polynomial of degree one in each argument, ie Q is a multiaffine polynomial.

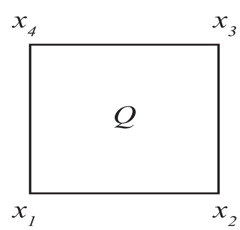

Figure 6: An elementary quadrilateral. A quad-equation $Q(x_1, x_2, x_3, x_4) = 0$.

Equations of type (86) are called *quad-equations*. The field variables x_i are assigned to four vertices of a quadrilateral as in a Figure 6. Equation (86) can be solved for each variable, and the solution is a rational function of the other three variables. A solution (x_1, x_2, x_3, x_4) of equation (86) is *singular* with respect to x_i if it also satisfy the equation $Q_{x_i}(x_1, x_2, x_3, x_4) = 0$.

Following [2] we consider the idea of integrability as consistency, see Figure 7. We assign six quad-equations to the faces of coordinate cube. The system is said to be *3D-consistent* if three values for x_{123} obtained from equations on right, back and top faces coincide for arbitrary initial data x, x_1, x_2, x_3.

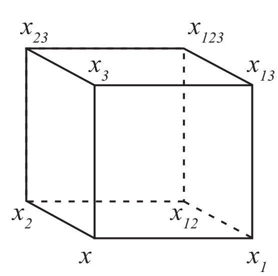

Figure 7: A 3D consistency.

Then, applying discriminant-like operators introduced in [2]

$$\delta_{x,y}(Q) = Q_x Q_y - Q Q_{xy}, \quad \delta_x(h) = h_x^2 - 2h h_{xx}, \tag{87}$$

one can make descent from the faces to the edges and then to the vertices of the cube: from a multiaffine polynomial $Q(x_1, x_2, x_3, x_4)$ to a biquadratic

polynomial $h(x_i,x_j) := \delta_{x_k,x_l}(Q(x_i,x_j,x_k,x_l))$ and further to a polynomial $P(x_i)=\delta_{x_j}(h(x_i,x_j))$ of degree up to four. By using of relative invariants of polynomials under fractional linear transformations, the formulae that express Q through biquadratic polynomials of three edges, were derived in [2]:

$$\frac{2Q_{x_1}}{Q} = \frac{h^{12}_{x_1}h^{34} - h^{14}_{x_1}h^{23} + h^{23}h^{34}_{x_3} - h^{23}_{x_3}h^{34}}{h^{12}h^{34} - h^{14}h^{23}}. \tag{88}$$

A biquadratic polynomial $h(x,y)$ is said to be *nondegenerate* if no polynomial in its equivalence class with respect to fractional linear transformations is divisible by a factor of the form $x-c$ or $y-c$, with $c = const$. A multiaffine function $Q(x_1,x_2,x_3,x_4)$ is said to be of *type* Q if all four of its accompanying biquadratic polynomials h^{jk} are nondegenerate. Otherwise, it is of *type* H. Previous notions were introduced in [2].

Lemma 6 *Given a biquadratic polynomial*

$$\begin{aligned} h(x_1,x_2) =& h_{22}x_1^2x_2^2 + h_{21}x_1^2x_2 + h_{12}x_1x_2^2 + h_{20}x_1^2 + h_{02}x_2^2 + h_{11}x_1x_2 \\ &+ h_{10}x_1 + h_{01}x_2 + h_{00} \end{aligned} \tag{89}$$

satisfying conditions of strong discriminant separability

$$\delta_{x_1}(h) = P(x_2), \quad \delta_{x_2}(h) = P(x_1) \tag{90}$$

where the discriminant $\delta_{x_i}(h)$ is given with (87) and $P(x_i)$ is a nonzero polynomial of degree up to four in the canonical form. Then, up to fractional linear transformations, h is equivalent to:

(A) *for* $P_A(x) = (k^2x^2-1)(x^2-1)$,

$$\begin{aligned} h =& -\frac{k^2}{4h_{20}}x_1^2x_2^2 + h_{20}(x_1^2+x_2^2) - \frac{1}{4h_{20}} \\ &\pm \frac{\sqrt{(2h_{20}-1)(2h_{20}+1)(2h_{20}-k)(2h_{20}+k)}}{2h_{20}}x_1x_2, \end{aligned} \tag{91}$$

(B) *for* $P_B(x) = x^2 - e^2$, $e \neq 0$

$$h = h_{20}(x_1^2+x_2^2) \pm \sqrt{1+4h_{20}^2}x_1x_2 + \frac{e^2}{4h_{20}}, \tag{92}$$

(C) *for* $P_C(x) = x^2$

$$\begin{aligned} h &= h_{20}x_1^2 + h_{11}x_1x_2 + h_{02}x_2^2, \quad h_{11}^2 - 4h_{20}h_{02} = 1, \\ h &= h_{22}x_1^2x_2^2 + h_{11}x_1x_2 + h_{00}, \quad h_{11}^2 - 4h_{22}h_{00} = 1, \end{aligned} \tag{93}$$

(D) *for* $P_D(x) = x$

$$h = -\frac{h_{11}}{2}(x_1 - x_2)^2 + \frac{1}{4h_{11}}(x_1 + x_2) - \frac{1}{32h_{11}^3}, \tag{94}$$

(E) *for* $P_E(x) = 1$

$$h = h_{20}(x_1 \pm x_2)^2 + h_{10}(x_1 \pm x_2) + h_{00}, \quad h_{10}^2 - 4h_{20}h_{00} = 1. \tag{95}$$

Proof. Proof is done by a straightforward calculation by solving the system of equations (90). □

With an additional assumption that the coefficients h_{ij} are quadratic functions of parameter α and with condition of discriminant separability, $h(x_1, x_2; \alpha)$ will actually represent discriminantly separable polynomial $\mathcal{F}(x_1, x_2, \alpha)$.

Notations for polynomials $P_A(x), ..., P_E(x)$ should not be mixed with functions P_1, P_2, P_3 from previous sections.

Now, we start with an arbitrary strongly discriminantly separable polynomial

$$\mathcal{F}(x_1, x_2, \alpha)$$

of degree two in each of three variables. We are going to construct an integrable quad-graph which corresponds to $\mathcal{F}(x_1, x_2, \alpha)$. In order to define such an integrable quad-graph we have to provide corresponding biquadratic polynomial $h = h(x_1, x_2)$ and a multiaffine polynomial $Q = Q(x_1, x_2, x_3, x_4)$.

The requirement that the discriminants of $h(x_1, x_2)$ do not depend on α, see [1], [2], will be satisfied if as a biquadratic polynomials $h(x_1, x_2)$ we take

$$\hat{h}(x_1, x_2) := \frac{\mathcal{F}(x_1, x_2, \alpha)}{\sqrt{P(\alpha)}}.$$

Proposition 10 *The polynomials*

$$\hat{h}(x_1, x_2) = \frac{\mathcal{F}(x_1, x_2, \alpha)}{\sqrt{P(\alpha)}} \tag{96}$$

satisfy

$$\delta_{x_1}(\hat{h}) = P(x_2), \quad \delta_{x_2}(\hat{h}) = P(x_1).$$

The list of biquadratic polynomials $\hat{h}(x_1, x_2)$ *having* $P_I(x)$, $I = A, B, C, D, E$ *as their discriminants, is:*

(A) *for* $P_A(x) = (k^2x^2 - 1)(x^2 - 1)$,

$$\begin{aligned}\hat{h}_A(x_1, x_2) =& \frac{(-k^2x_1^2 - k^2x_2^2 + 1 + k^2x_1^2x_2^2)\alpha^2 + (1 - k^2)x_1x_2\alpha}{2\sqrt{(k^2\alpha^2 - 1)(\alpha^2 - 1)}} \\ &+ \frac{x_1^2 + x_2^2 - k^2x_1^2x_2^2 - 1}{4\sqrt{(k^2\alpha^2 - 1)(\alpha^2 - 1)}}.\end{aligned}$$

Remark here that $\hat{h}_A(x_1,x_2) = h$ from (91) for

$$h_{20} = \frac{-\sqrt{k^2\alpha^2-1}}{2\sqrt{\alpha^2-1}}.$$

(B) *for $P_B(x) = x^2 - e^2$, $e \neq 0$,*

$$\hat{h}_B(x_1,x_2) = \frac{x_1x_2\alpha + \frac{e}{2}(x_1^2+x_2^2+\alpha^2-e^2)}{\sqrt{\alpha^2-e^2}},$$

and $\hat{h}_B(x_1,x_2) = h$ from (92) for

$$h_{20} = \frac{e}{2\sqrt{\alpha^2-e^2}}.$$

(C) *for $P_C(x) = x^2$,*

$$\hat{h}_{C1}(x_1,x_2) = \frac{\lambda x_1^2\alpha^2 + \mu x_1x_2\alpha + \nu x_2^2}{\alpha}, \quad \mu^2 - 4\lambda\nu = 1,$$

$$\hat{h}_{C2}(x_1,x_2) = \frac{\lambda x_1^2x_2^2\alpha^2 + \mu x_1x_2\alpha + \nu}{\alpha}, \quad \mu^2 - 4\lambda\nu = 1.$$

(D) *for $P_D(x) = x$,*

$$\hat{h}_D(x_1,x_2) = \frac{-(x_1x_2 + x_1\alpha + x_2\alpha) + \frac{1}{4}(x_1^2+x_2^2+\alpha^2)}{2\sqrt{\alpha}},$$

and $\hat{h}_D(x_1,x_2) = h$ from (94) for

$$h_{11} = \frac{-1}{2\sqrt{\alpha}}.$$

(E) *for $P_E(x) = 1$,*

$$\hat{h}_{E1}(x_1,x_2) = \lambda(x_1+x_2+\alpha)^2 + \mu(x_1+x_2+\alpha) + \nu, \quad \mu^2 - 4\lambda\nu = 1,$$
$$\hat{h}_{E2}(x_1,x_2) = \lambda(-x_1+x_2+\alpha)^2 + \mu(-x_1+x_2+\alpha) + \nu, \quad \mu^2 - 4\lambda\nu = 1,$$
$$\hat{h}_{E3}(x_1,x_2) = \lambda(x_1-x_2+\alpha)^2 + \mu(x_1-x_2+\alpha) + \nu, \quad \mu^2 - 4\lambda\nu = 1,$$
$$\hat{h}_{E4}(x_1,x_2) = \lambda(x_1+x_2-\alpha)^2 + \mu(x_1+x_2-\alpha) + \nu, \quad \mu^2 - 4\lambda\nu = 1.$$

Notice that polynomials $\hat{h}_{C1}$, $\hat{h}_{C2}$, $\hat{h}_{Ei}$ may be rewritten as (93) and (95):

$$\hat{h}_{C1}(x_1,x_2) = \lambda\alpha x_1^2 + \mu x_1x_2 + \frac{\nu}{\alpha}x_2^2 = \lambda' x_1^2 + \mu x_1x_2 + \nu' x_2^2, \quad \mu^2 - 4\lambda'\nu' = 1$$

$$\hat{h}_{C2}(x_1,x_2) = \lambda\alpha x_1^2 + \mu x_1x_2 + \frac{\nu}{\alpha}x_2^2 = \lambda' x_1^2 + \mu x_1x_2 + \nu' x_2^2, \quad \mu^2 - 4\lambda'\nu' = 1$$

$$\begin{aligned}\hat{h}_{E1}(x_1,x_2) &= \lambda(x_1+x_2)^2 + 2\alpha\lambda(x_1+x_2) + \lambda\alpha^2 + \mu(x_1+x_2) + \mu\alpha + \nu \\ &= \lambda(x_1+x_2)^2 + (\mu+2\alpha\lambda)(x_1+x_2) + \lambda\alpha^2 + \mu\alpha + \nu \\ &= \lambda(x_1+x_2)^2 + \mu'(x_1+x_2) + \nu', \quad \mu'^2 - 4\lambda\nu' = \mu^2 - 4\lambda\nu = 1,\end{aligned}$$

and, in the same way for $\hat{h}_{E2} - \hat{h}_{E4}$.

The final problem of reconstructing *quad-equations of type Q* which correspond to strongly discriminantly separable polynomials obtained in Theorem 4 is done with a help of formulae (88) and polynomials h^{ij} replaced by $\hat{h}^{ij}$.

Here we take

$\hat{h}^{12} = \hat{h}_I(x_1, x_2; \alpha)$, $\hat{h}^{23} = \hat{h}_I(x_2, x_3; \beta)$, $\hat{h}^{34} = \hat{h}_I(x_3, x_4; \alpha)$, $\hat{h}^{14} = \hat{h}_I(x_1, x_4; \beta)$,

for $I = A, B, C, D, E$.

Theorem 5 *Quad-equations of type Q that correspond to the biquadratic polynomials*

$$\hat{h}(x_1, x_2; \alpha) = \frac{\mathcal{F}(x_1, x_2, \alpha)}{\sqrt{P_I(\alpha)}}$$

are given in the following list:

$$\begin{aligned}
\hat{Q}_A =& \frac{\beta\sqrt{P_A(\alpha)} + \alpha\sqrt{P_A(\beta)}}{k^2\alpha^2\beta^2 - 1}\sqrt{\frac{\alpha^2 - 1}{k^2\alpha^2 - 1}}\sqrt{\frac{\beta^2 - 1}{k^2\beta^2 - 1}}(k^2 x_1 x_2 x_3 x_4 + 1) \\
&+ \sqrt{\frac{\beta^2 - 1}{k^2\beta^2 - 1}}(x_1 x_2 + x_3 x_4) + \sqrt{\frac{\beta^2 - 1}{k^2\beta^2 - 1}}(x_1 x_4 + x_2 x_3) \\
&+ \frac{\beta\sqrt{P_A(\alpha)} + \alpha\sqrt{P_A(\beta)}}{k^2\alpha^2\beta^2 - 1}(x_1 x_3 + x_2 x_4) = 0, \\
\hat{Q}_B =& \sqrt{\alpha^2 - e^2}(x_1 x_2 + x_3 x_4) + \sqrt{\beta^2 - e^2}(x_1 x_4 + x_2 x_3) \\
&+ \frac{\alpha\sqrt{\beta^2 - e^2} + \beta_1\sqrt{\alpha_1^2 - e^2}}{e}(x_1 x_3 + x_2 x_4) \\
&- \frac{\sqrt{\beta^2 - e^2}\sqrt{\alpha^2 - e^2}(\alpha\sqrt{\beta^2 - e^2} + \beta\sqrt{\alpha^2 - e^2})}{e} = 0, \\
\hat{Q}_C =& (\alpha - \frac{1}{\alpha})(x_1 x_2 + x_3 x_4) + (\beta - \frac{1}{\beta})(x_1 x_4 + x_2 x_3) \\
&- (\alpha\beta - \frac{1}{\alpha\beta})(x_1 x_3 + x_2 x_4) = 0, \\
\hat{Q}_D =& \sqrt{\alpha}(x_1 - x_4)(x_2 - x_3) + \sqrt{\beta}(x_1 - x_2)(x_4 - x_3) \\
&- \sqrt{\alpha}\sqrt{\beta}(\sqrt{\alpha} + \sqrt{\beta})(x_1 + x_2 + x_3 + x_4) \\
&+ \sqrt{\alpha}\sqrt{\beta}(\sqrt{\alpha} + \sqrt{\beta})(\alpha + \sqrt{\alpha\beta} + \beta) = 0, \\
\hat{Q}_E =& \alpha(x_1 - x_4)(x_2 - x_3) + \beta(x_1 - x_2)(x_4 - x_3) - \alpha\beta(\alpha + \beta) = 0.
\end{aligned}$$

Proof. From (88) we get that when

$$\hat{h}^{12} = \hat{h}_A(x_1, x_2; \alpha),\ \hat{h}^{23} = \hat{h}_A(x_2, x_3; \beta),$$

$$\hat h^{34} = \hat h_A(x_3, x_4; \alpha),\ \hat h^{14} = \hat h_A(x_1, x_4; \beta),$$

corresponding Q is

$$\hat Q_A = a_1 x_1 x_2 x_3 x_4 + a_6(x_1x_2 + x_3x_4) + a_8(x_1x_4 + x_2x_3) + a_{10}(x_1x_3 + x_2x_4) + a_{16}$$

with

$$\begin{aligned}
a_1 &= \frac{a_{10} k^2 \sqrt{P_A(\alpha)}\sqrt{P_A(\beta)}}{k^2\alpha^2\beta^2} \\
a_6 &= \frac{a_{10}\sqrt{P_A(\alpha)}(k^2\alpha^2\beta^2 - 1)}{(\alpha\sqrt{P_A(\beta)} + \beta\sqrt{P_A(\alpha)})(k^2\alpha^2 - 1)} \\
a_8 &= \frac{a_{10}\sqrt{\beta^2 - 1}(\beta\sqrt{P_A(\alpha)} - \alpha\sqrt{P_A(\beta)})}{\sqrt{k^2\beta^2 - 1}(\alpha^2 - \beta^2)} \\
a_{16} &= \frac{a_{10}\sqrt{P_A(\alpha)}\sqrt{P_A(\beta)}}{k^2\alpha^2\beta^2}.
\end{aligned}$$

Recall here that $P_A(x) = (k^2x^2 - 1)(x^2 - 1)$. Choosing

$$a_{10} = \frac{\beta\sqrt{P_A(\alpha)} + \alpha\sqrt{P_A(\beta)}}{k^2\alpha^2\beta^2 - 1}$$

we finally get $\hat Q_A$.

In the same way for other choices $\hat h^{ij} = \hat h_I(x_i, x_j)$, $I = B, C, D, E$ from (88) we get expressions $\hat Q_B$-$\hat Q_E$.

$\square$

Remark 2 *The list of biquadratic polynomials from Lemma 6 contains the list of h's obtained in [2]. More precisely replacing*

$$h_{20} = \frac{1}{2\alpha}$$

into (91) it becomes

$$h = \frac{1}{2\alpha}(k^2\alpha^2 x_1^2 x_2^2 + 2\sqrt{P_A(\alpha)}x_1x_2 - x_1^2 - x_2^2 + \alpha^2). \tag{97}$$

Next, with a choice

$$h_{20} = \frac{\alpha}{1 - \alpha^2}$$

(92) turns into

$$h = \frac{\alpha}{1 - \alpha^2}(x_1^2 + x_2^2) - \frac{1 + \alpha^2}{1 - \alpha^2}x_1x_2 + \frac{e^2(1 - \alpha^2)}{4\alpha}. \tag{98}$$

Finally, for

$$h_{11} = -\frac{1}{2\alpha}$$

from (94) we get

$$h = \frac{1}{4\alpha}(x_1 - x_2)^2 - \frac{\alpha}{2}(x_1 + x_2) + \frac{\alpha^3}{4}. \tag{99}$$

The expressions for h in [2] for $P(x) = x^2$ and $P(x) = 1$ are the same as corresponding h in Lemma 6.

In the same manner, the list obtained in Theorem 5 corresponds with the list of multiaffine equations of type Q obtained in [2]:

$$\begin{aligned} &sn(\alpha)sn(\beta)sn(\alpha+\beta)(k^2x_1x_2x_3x_4 + 1) - sn(\alpha)(x_1x_2 + x_3x_4) \\ &- sn(\beta)(x_1x_4 + x_2x_3) + sn(\alpha+\beta)(x_1x_3 + x_2x_4) = 0, \end{aligned} \tag{100}$$

$$\begin{aligned} &(\alpha - \alpha^{-1})(x_1x_2 + x_3x_4) + (\beta - \beta^{-1})(x_1x_4 + x_2x_3) - (\alpha\beta - \alpha^{-1}\beta^{-1})(x_1x_3 + x_2x_4) \\ &+ \frac{\delta}{4}(\alpha - \alpha^{-1})(\beta - \beta^{-1})(\alpha\beta - \alpha^{-1}\beta^{-1}) = 0, \end{aligned} \tag{101}$$

$$\begin{aligned} &\alpha(x_1 - x_4)(x_2 - x_3) + \beta(x_1 - x_2)(x_4 - x_3) - \alpha\beta(\alpha+\beta)(x_1 + x_2 + x_3 + x_4) \\ &+ \alpha\beta(\alpha+\beta)(\alpha^2 + \alpha\beta + \beta^2) = 0, \end{aligned} \tag{102}$$

$$\alpha(x_1 - x_4)(x_2 - x_3) + \beta(x_1 - x_2)(x_4 - x_3) - \delta\alpha\beta(\alpha+\beta) = 0. \tag{103}$$

To see the correspondence of $\hat{Q}_A = 0$ and (100) we need to write down the form of (100) one step before the final solution, which is

$$\alpha\beta\gamma(k^2x_1x_2x_3x_4 + 1) + \alpha(x_1x_2 + x_3x_3) + \beta(x_1x_4 + x_2x_3) + \gamma(x_1x_3 + x_2x_4) = 0 \tag{104}$$

where

$$\gamma = \frac{\alpha\sqrt{P_A(\beta)} + \beta\sqrt{P_A(\alpha)}}{k^2\alpha^2\beta^2 - 1}.$$

Then, (100) is obtained from (104) with $\alpha \mapsto sn(\alpha)$, $\sqrt{P_A(\alpha)} \mapsto sn'(\alpha)$ and similarly for β.
If we replace parameters α, β in (100) with

$$\alpha \mapsto \sqrt{\frac{\alpha^2 - 1}{k^2\alpha^2 - 1}}, \quad \beta \mapsto \sqrt{\frac{\beta^2 - 1}{k^2\beta^2 - 1}}$$

then the correspondence of $\hat{Q}_A$ and (104) is obvious.

The correspondence of $\hat{Q}_B$ and (101) with $\delta = e^2$ is achieved with a change

$$\alpha \mapsto \frac{\alpha - \sqrt{\alpha^2 - e^2}}{e}, \ \beta \mapsto \frac{\beta - \sqrt{\beta^2 - e^2}}{e}.$$

Notice here that with the last change we get

$$\alpha - \alpha^{-1} \mapsto \frac{-2\sqrt{\alpha_1^2 - e^2}}{e}$$

and similarly for β*.*

Finally, a change of parameters

$$\alpha \mapsto \sqrt{\alpha}, \quad \beta \mapsto \sqrt{\beta}$$

brings $\hat{Q}_D = 0$ *into corresponding (102).*

Equation $Q(x_1, x_2, x_3, x_4) = 0$ that corresponds to polynomial $\mathcal{F}$ given with (84) is obtained in [1]:

$$\begin{aligned}
&a_0x_1x_2x_3x_4 + a_1(x_1x_2x_3 + x_1x_2x_4 + x_1x_3x_4 + x_2x_3x_4) + a_2(x_1x_3 + x_2x_4) \\
&+ \bar{a}_2(x_1x_2 + x_3x_4) + \tilde{a}_2(x_1x_4 + x_2x_3) + a_3(x_1 + x_2 + x_3 + x_4) + a_4 = 0,
\end{aligned}$$

where coefficients are

$$\begin{gathered}
a_0 = a + b, \quad a_1 = -\beta a - \alpha b, \quad a_2 = \beta^2 a + \alpha^2 b, \\
\bar{a}_2 = \frac{ab(a+b)}{2(\alpha - \beta)} + \beta^2 a - (2\alpha^2 - \frac{g_2}{4})b, \\
\tilde{a}_2 = \frac{ab(a+b)}{2(\beta - \alpha)} + \alpha^2 b - (2\beta^2 - \frac{g_2}{4})a, \\
a_3 = \frac{g_3}{2}a_0 - \frac{g_2}{4}a_1, \quad a_4 = \frac{g_2^2}{16}a_0 - g_3 a_1,
\end{gathered}$$

and

$$a^2 = 4\alpha^3 - g_2\alpha - g_3, \quad b^2 = 4\beta^3 - g_2\beta - g_3.$$

Previous calculations establish a relation between strongly discriminantly separable polynomials in three variables of degree two in each and biquadratic polynomials from the theory of quad-graphs which correspond to *the diagonal elements* of table on page 11, paper [2]. Remaining three *off–diagonal cases* are connected with symmetrically discriminantly separable polynomials (see [15] and Introduction for the definition of symmetrically discriminantly separable polynomials). The next Remark is devoted to that case.

Remark 3 *The three remaining, off-diagonal, biquadratic polynomials obtained in the list in [2] are:*

$$\begin{aligned}
h_1(x_1, x_2) &= \alpha x_2^2 \pm x_1x_2 + \frac{1}{4\alpha}, \\
h_2(x_1, x_2) &= \pm\frac{1}{4}(x_2 - \alpha)^2 \mp x_1, \\
h_3(x_1, x_2) &= \lambda x_2^2 + \mu x_2 + \nu, \quad \mu^2 - 4\lambda\nu = 1,
\end{aligned}$$

with the corresponding discriminants $\delta_{x_i}(h_j)$, $i = 1, 2$, $j = 1, 2, 3$:

$$\begin{aligned} &\delta_{x_1}(h_1) = x_2^2, \quad \delta_{x_2}(h_1) = x_1^2 - 1,\\ &\delta_{x_1}(h_2) = 1, \quad \delta_{x_2}(h_2) = x_1,\\ &\delta_{x_1}(h_3) = 0, \quad \delta_{x_2}(h_3) = 1. \end{aligned}$$

The polynomials h_1, h_2, h_3 obviously do not correspond to strongly discriminantly separable polynomials. However, there are symmetrically discriminantly separable polynomials which correspond to these cases. (See the Introduction for the definition of symmetrically discriminantly separable polynomials.) With correlation (96), we get for the case of h_1:

$$\mathcal{F}_1(x_1, x_2, x_3) = x_2^2 x_3^2 \pm x_1 x_2 x_3 + \frac{1}{4}$$

with

$$\begin{aligned} \mathcal{D}_{x_1}(\mathcal{F}_1) &= x_2^2 x_3^2,\\ \mathcal{D}_{x_2}(\mathcal{F}_1) &= (x_1^2 - 1) x_3^2,\\ \mathcal{D}_{x_3}(\mathcal{F}_1) &= (x_1^2 - 1) x_2^2. \end{aligned}$$

For the case h_2, let us consider the next polynomial:

$$\mathcal{F}_2(x_1, x_2, x_3) = \pm\frac{(x_2 - x_3)^2}{4} \mp x_1$$

with

$$\begin{aligned} \mathcal{D}_{x_1}(\mathcal{F}_2) &= 1,\\ \mathcal{D}_{x_2}(\mathcal{F}_2) &= x_1,\\ \mathcal{D}_{x_3}(\mathcal{F}_2) &= x_1. \end{aligned}$$

Finally, for the case h_3 we take:

$$\mathcal{F}_3(x_1, x_2, x_3) = \lambda x_3^2 + \mu x_3 + \nu, \quad \mu^2 - 4\lambda\nu = 1$$

with

$$\begin{aligned} \mathcal{D}_{x_1}(\mathcal{F}_3) &= 0,\\ \mathcal{D}_{x_2}(\mathcal{F}_3) &= 0,\\ \mathcal{D}_{x_3}(\mathcal{F}_3) &= 1. \end{aligned}$$

Let us underline that the correspondences established in the current Section, provide a two way-link between strongly discriminantly separable polynomials and quad-graphs. As a benefit, we get a geometric interpretation of biquadratic polynomials h from the theory of quad-graphs relating them to pencils of conics. Moreover, in the case [A], coded $(1, 1, 1, 1)$, when a polynomial P has four distinct zeros, the biquadratic polynomials of quad-graphs correspond to the two-valued Buchstaber-Novikov groups associated with elliptic curves, i.e. those defined on $\mathbb{CP}^1$.

6 Some more examples of Kowalevski-type systems

We can construct examples of systems of Kowalevski type using procedure described in Section 2.1, either by choosing functions p_i and r_i which satisfy

$$(\sqrt{p_1 r_2} - \sqrt{p_2 r_1})^2 = (x_1 - x_2)^2$$

with

$$f_i = \sqrt{r_i} r + \sqrt{p_i}\gamma_3,\ i = 1, 2$$

or by applying the method from Lemma 2 on the functions of type

$$f_i = (x_i - \alpha_i)^{m_i} r + (x_i - \beta_i)^{n_i}\gamma_3,\ i = 1, 2.$$

Example 2 *In this example we will find the equations of motion for a system that reduces to (15), (16) and (17) with functions f_i nonsymmetric on x_i. For*

$$p_1 = (x_1 + x_2)^2, \quad p_2 = (2x_2)^2, \quad r_1 = 1, \quad r_2 = 1,$$

and

$$f_1 = (x_1 + x_2)\gamma_3 + r, \quad f_2 = 2x_2\gamma_3 + r$$

we get two systems of equations which can be solved in terms of P_i, P_{ij} functions following Kowalevski's procedure as proposed in Section 2.3. We start from assumption that system (15) is:

$$\begin{aligned} 2\dot{x}_1 &= -\imath((x_1 + x_2)r + \gamma_3) \\ 2\dot{x}_2 &= \imath(2x_2 r + \gamma_3) \\ \dot{e}_1 &= -me_1 \\ \dot{e}_2 &= me_2. \end{aligned} \tag{105}$$

Let us write down possible first integrals in the form (43). Now, reducing

$$(p_2 e_1 + p_1 e_2 + E)(r_2 e_1 + r_1 e_2 + G) - (F - q_2 e_1 - q_1 e_2)^2 = 0$$

to relation (17) we get expressions for E, F, G.

$$\begin{aligned} &(p_2 e_1 + p_1 e_2 + E)(r_2 e_1 + r_1 e_2 + G) - (F - q_2 e_1 - q_1 e_2)^2 \\ &= (x_1 - x_2)^2 e_1 e_2 + (4x_2^2 G + 4x_2 F + E)e_1 + ((x_1 + x_2)^2 G + 2(x_1 + x_2)F + E)e_2 \\ &+ EG - F^2 = 0. \end{aligned}$$

By equating coefficients with e_i in the last relation and (17) with A, C, P as in (26), (27) we get the solutions

$$\begin{aligned} E_1 &= 4x_2^2 x_1 + 2x_2^3 + \frac{g_2}{2}x_2 - \frac{g_3}{2} \\ F_1 &= F = -3x_1 x_2 - 2x_2^2 - \frac{g_2}{2} \\ G_1 &= 2x_1 + 2x_2, \end{aligned} \tag{106}$$

and values E_2, F_2, G_2 which we will not specify at the moment. With choice of E_1, F_1, G_1 we get system of differential equations with constant parameter g_2 for which

$$\begin{aligned} r^2 &= 4x_2^2x_1 + 2x_2^3 + \frac{g_2}{2}x_2 - \frac{g_3}{2} + 4x_2^2e_1 + (x_1+x_2)^2e_2 \\ e_1e_2 &= k^2 \end{aligned} \tag{107}$$

are the first integrals and that is integrated on four-dimensional submanifold given with equations

$$\begin{aligned} r\gamma_3 &= -3x_1x_2 - 2x_2^2 - \frac{g_2}{4} - 2x_2e_1 - (x_1+x_2)e_2 \\ \gamma_3^2 &= 2x_1 + 2x_2 + e_1 + e_2. \end{aligned} \tag{108}$$

Finally, system of differential equations that satisfies these assumptions is

$$\begin{aligned} 2\dot{x}_1 &= -\imath((x_1+x_2)r+\gamma_3) \\ 2\dot{x}_2 &= \imath(2x_2r+\gamma_3) \\ \dot{e}_1 &= -me_1 \\ \dot{e}_2 &= me_2 \\ \dot{r} &= \frac{32\imath x_2^2r + 16\imath x_2\gamma_3 - 16x_2^2m}{8r}e_1 \\ &+ \frac{4\imath x_2^2r + 4mx_1^2 + 8mx_1x_2 - 4\imath rx_1^2 - 4x_2^2m}{8r}e_2 \\ &+ \imath\frac{24x_2^2x_1r + 16x_2x_1\gamma_3 + 16x_2^3r + 4x_2^2\gamma_3 + 2g_2x_2r + g_2\gamma_3}{8r} \\ \dot{\gamma}_3 &= \frac{m(e_2-e_1) + \imath r(x_2-x_1)}{2\gamma_3}. \end{aligned} \tag{109}$$

Here with m we denote

$$\begin{aligned} m = &\frac{\imath}{(\gamma_3(x_1+x_2)+r)^2e_2 - (r+2x_2\gamma_3)^2e_1}\big(4r\gamma_3(x_1-x_2)(\gamma_3(x_1+x_2)+r)e_2 \\ &- 8\gamma_3(2x_2r+\gamma_3)(r+2x_2\gamma_3)e_1 - 16\gamma_3^3x_2x_1 - 4\gamma_3^3x_2^2 - \gamma_3^3g_2 - 16\gamma_3^2x_2^3r \\ &- 2\gamma_3^2g_2x_2r + 4r^3x_1 - 12x_1\gamma_3^2r - 12x_2r^2x_1\gamma_3 - 20x_2^2r^2\gamma_3 - 4x_2\gamma_3^2r \\ &- 24\gamma_3^2x_2^2x_1r - 4r^3x_2\big). \end{aligned}$$

Relations (107) and (108) lead us to the relation

$$\begin{aligned} &(x_1-x_2)^2e_1e_2 + (2x_1^3 - \frac{g_2}{2}x_1 - \frac{g_3}{2})e_2 + (2x_2^3 - \frac{g_2}{2}x_2 - \frac{g_3}{2})e_1 \\ &- x_1^2x_2^2 - \frac{g_2}{2}x_1x_2 - g_3(x_1+x_2) - \frac{g_2^2}{16} = 0, \end{aligned}$$

which gives a strongly discriminantly separable polynomial in the form (85) with the procedure described in Theorem 1.

Example 3 *In the same way is in previous example we can find system of Kowalevski type for*

$$f_i = \frac{r}{x_i} + \gamma_3 x_j^2, \quad (i,j) = c.p.(1,2). \tag{110}$$

Here for p_i, r_i we take

$$p_1 = x_1^4, \quad p_2 = x_2^4, \quad r_1 = \frac{1}{x_2^2}, \quad r_2 = \frac{1}{x_1^2},$$

Again reducing

$$(p_2e_1 + p_1e_2 + E)(r_2e_1 + r_1e_2 + G) - (F - q_2e_1 - q_1e_2)^2 = 0$$

to relation (17) with A, C, P as in (26), (27) we get

$$\begin{aligned}
E_1 &= -\frac{g_3}{2}(x_1^2 + x_2^2) - g_3x_2x_1 - \frac{g_2}{2}x_2x_1^2 - \frac{g_2}{2}x_1x_2^2\\
F_1 &= \frac{2g_3(x_1 + x_2) + 4x_1^2x_2^2 + x_1x_2g_2}{4x_1x_2}\\
G_1 &= -\frac{g_3}{2x_1^2x_2^2},
\end{aligned}$$

and

$$\begin{aligned}
E_2 &= -\frac{g_2x_1^2x_2^2(x_1 + x_2) + g_2x_1x_2(x_1^3 + x_2^3) + g_3(x_1^2 + x_2^2)^2 - 8x_1^3x_2^3(x_1 + x_2)}{2(x_1 - x_2)^2}\\
F_2 &= -\frac{24x_1^3x_2^3 + 4x_1^2x_2^2(x_1^2 + x_2^2) - 2g_2x_1^2x_2^2 - 2g_3x_1x_2(x_1 + x_2)}{4x_1x_2(x_1 - x_2)^2}\\
&\quad + \frac{2g_3(x_1^3 + x_2^3) + 3g_2x_1x_2(x_1^2 + x_2^2)}{4x_1x_2(x_1 - x_2)^2}\\
G_2 &= \frac{(x_1 + x_2)(8x_1^2x_2^2 - g_3(x_1 + x_2) - 2g_2x_1x_2)}{2x_1^2x_2^2(x_1 - x_2)^2}.
\end{aligned}$$

By differentiating relations

$$r^2 = E_1 + p_2e_1 + p_1e_2$$

and

$$\gamma_3^2 = G_1 + r_2e_1 + r_1e_2$$

together with (110) we get equations

$$\begin{aligned}
\dot{r} &= \frac{4\imath x_2^4r - 2x_2^5mx_1 + 4\imath x_2^6\gamma_3x_1}{4rx_1x_2}e_1 + \frac{-4\imath x_1^4r + 2x_1^5mx_2 - 4\imath x_1^6\gamma_3x_2}{4rx_1x_2}e_2\\
&\quad - \frac{(r + x_1x_2^2\gamma_3 + x_1^2x_2\gamma_3)(x_2g_3 + x_2x_1g_2 + x_1g_3)(x_2 - x_1)\imath}{4rx_1x_2}\\
\dot{\gamma}_3 &= \frac{-2mx_2^3x_1 + 2\imath x_2^2r + 2\imath x_2^3x_1^2\gamma_3}{4\gamma_3x_1^3x_2^3}e_1 + \frac{2mx_1^3x_2 - 2\imath x_1^2r - 2\imath x_1^3x_2^2\gamma_3}{4\gamma_3x_1^3x_2^3}e_2\\
&\quad + \frac{\imath\gamma_3(x_1 - x_2)x_2x_1g_3}{4\gamma_3x_1^3x_2^3}.
\end{aligned}$$

Finally, by substituting these expressions into the relation obtained by differentiating

$$r\gamma_3 = F_1 - q_2 e_1 - q_1 e_2$$

we get

$$\begin{aligned}
m =& \frac{2x_1^2(r + x_1^2\gamma_3 x_2)(r^2 + 2x_1^2\gamma_3 x_2 r + x_1 r x_2^2\gamma_3 + 2\gamma_3^2 x_2^2 x_1^4)e_2}{x_1 x_2(x_1^2(r + x_1^2 x_2\gamma_3)^2 e_2 - x_2^2(r + x_2^2 x_1\gamma_3)^2 e_1)} \\
&- \frac{2x_2^2(r + x_2^2\gamma_3 x_1)(r^2 + 2x_2^2\gamma_3 x_1 r + x_2 r x_1^2\gamma_3 + 2\gamma_3^2 x_1^2 x_2^4)e_1}{x_1 x_2(x_1^2(r + x_1^2 x_2\gamma_3)^2 e_2 - x_2^2(r + x_2^2 x_1\gamma_3)^2 e_1)} \\
&- \frac{(x_1 - x_2)\gamma_3^2 x_2^2 x_1^2}{x_1 x_2(x_1^2(r + x_1^2 x_2\gamma_3)^2 e_2 - x_2^2(r + x_2^2 x_1\gamma_3)^2 e_1)} \\
&(x_1^3 x_2^2 g_2\gamma_3 + x_1^3 x_2 g_3\gamma_3 + x_1^2 x_2^3 g_2\gamma_3 + 2x_1^2 x_2^2 g_3\gamma_3 + 2x_1^2 x_2^2 r + x_1 x_2^3 g_3\gamma_3 \\
&+ x_1 x_2 g_2 r + g_3 r(x_1 + x_2)).
\end{aligned}$$

This way, we got system of Kowalevski type with two constant parameters g_2, g_3.

Remark 4 *The systems of equations described in previous Examples all reduce to (25)-(27). Thus the procedure of integration may be completed in the way shown in Section (2.2). We get x_1 and x_2 in terms of $\wp$-function and then, from the first integrals (43) we can express other variables in terms of functions P_i and P_{ij}, $i, j = 1, 2, 3$, either by use of the Kötter trick, or following Kowalevski-type procedure. How to express the functions P_i and P_{ij}, $i, j = 1, 2, 3$ in terms of theta-functions, we are going to demonstrate in the following Appendix.*

7 Appendix: Theta-functions formulae

Here we will give explicit formulae for the functions

$$P_i,\, P_{ij},\, i,j = 1, ..., 5,$$

defined in (29, 30), in terms of genus two theta-functions. More about derivation of these formulae, one can find in [5], [24].

First, for reader's sake, we give a very brief introduction to a modern exposition of theta-function theory. More on this topic one may found in [17] or [27] for example.

Let Γ be a compact Riemann surface of an algebraic function $w = w(z)$ that satisfies

$$R(w, z) = w^n + a_1(z)w^{n-1} + ... + a_n(z) = 0,$$

where $a_i(z)$ are polynomials in z with complex coefficients. As a special cases, we single out hyperelliptic curves which can be given with an equation

$$w^2 = P_{2g+1}(z), \tag{111}$$

or, with

$$w^2 = P_{2g+2}(z), \tag{112}$$

where $P_{2g+1}(z)$ and $P_{2g+2}(z)$ are polynomials without multiplied roots of degree respectively $2g + 1$, $2g + 2$. Hyperelliptic curves described with (111) or (112) are of genus g, $g > 1$. When $g = 1$ corresponding curve is called elliptic.

We can fix a canonical basis of cycles $a_1, b_1, ..., a_g, b_g$ on a surface Γ of genus g such that

$$a_i \circ a_j = b_i \circ b_j = 0, \quad a_i \circ b_j = \delta_{ij}.$$

For a given canonical homology basis, we choose the corresponding g-dimensional basis of normalized holomorphic differentials ω_i, $i = 1, ..., g$ such that

$$\oint_{a_i} \omega_j = \delta_{ij}.$$

The b-period matrix is then defined to be

$$\tau_{jk} = \oint_{b_j} \omega_k \tag{113}$$

and it is a so-called Riemann matrix. Recall that the Riemann matrix is a symmetric square matrix whose imaginary part is positive definite. The integrals specified above are called normal elementary integrals of the first kind. A

general integral of the first kind is their linear combination.

The basic vectors e_k, $k = 1, ..., g$ of $\mathbb{C}^g$ and vectors τ_k which are the columns of the matrix (113) generate a lattice

$$\Lambda = \sum_{k=1}^{g} m_k e_k + n_k \tau_k, \quad m_k, n_k \in \mathbb{Z}.$$

The quotient space

$$J(\Gamma) = \mathbb{C}^g / \Lambda$$

is topologically a $2g$-dimensional torus, and it is called the Jacobian variety of Γ.

The Abel map $A : \Gamma \to J(\Gamma)$, $A(P) = (A_1(P), ..., A_g(P))$ is then defined by

$$A_k(P) = \int_{P_0}^{P} \omega_k, \quad k = 1, ..., g,$$

and P_0 is a fixed initial point on the Riemann surface. If $g > 0$, the Abel map is embedding of Γ into $J(\Gamma)$.

The Abel map is extended to arbitrary divisors $\mathcal{D} = \sum k_j P_j$ as follows

$$A(\mathcal{D}) = \sum k_j A(P_j).$$

Now, we can state Abel's theorem: *Divisor $\mathcal{D}$ of degree 0 is the divisor of a meromorphic function if and only if* $A(\mathcal{D}) \equiv 0$, (where $\equiv$ denotes matching up to the lattice Λ). As another formulation of the Abel theorem one can say that two divisors $\mathcal{D}$ and $\mathcal{D}'$ are linearly equivalent if and only if $deg\mathcal{D} = deg\mathcal{D}'$ and $A(\mathcal{D}) \equiv A(\mathcal{D}')$.

The Riemann theta function associated to the matrix τ is the following function of g variables:

$$\theta(z_1, ..., z_g) = \theta(z) = \theta(z|\tau) = \sum_{m \in \mathbb{Z}^g} e^{2\pi i (z,m) + \pi i (\tau m, m)},$$

where $m = (m_1, ..., m_g)$, $(z, m) = z_1 m_1 + z_2 m_2 + ... + z_g m_g$ and
$(\tau m, m) = \tau_{11} m_1 m_1 + \tau_{12} m_1 m_2 + ... + \tau_{gg} m_g m_g$.

When the matrix τ is fixed, then $\theta(z|\tau)$ is denoted simply with $\theta(z)$. It has the following properties:

$$\begin{aligned} \theta(z|\tau) &= \theta(-z|\tau), \\ \theta(z + e_k|\tau) &= \theta(z|\tau), \\ \theta(z + \tau_k|\tau) &= e^{-2\pi i z_k - \pi i \tau_{kk}} \theta(z|\tau), \\ \theta(z + m + \tau n|\tau) &= e^{-2\pi i (n,z) - \pi i (\tau n, n)} \theta(z|\tau), \quad m, n \in \mathbb{Z}^g. \end{aligned} \tag{114}$$

Theta function with characteristics is introduced with

$$\theta[\alpha,\beta](z)=\sum_{m\in\mathbb{Z}^g} e^{2\pi i(z+\alpha,m+\beta)+\pi i(\tau(m+\beta),m+\beta)}$$

with $\alpha=(\alpha_1,...,\alpha_g)$, $\beta=(\beta_1,...,\beta_g)$. Characteristics $[\alpha,\beta]$ with all coordinates α_i,β_j equals to 0 or $\frac{1}{2}$ are called half-periods, even for $4(\alpha,\beta)\equiv 0\,(mod\,2)$ and odd half-period in opposite case.

Consider now for a fixed vector $e=(e_1,...,e_g)\in\mathbb{C}^g$ the function

$$F(P)=\theta(A(P)-e)$$

with $A(P)$- the Abel map with a base point P_0. For $F(P)$ not identically zero, the equation

$$\theta(A(P)-e)=0$$

has g zeros $P_1,...,P_g$. Then on $J(\Gamma)$, the relation holds:

$$A(P_1,...,P_g)\equiv e-\mathcal{K},$$

where $\mathcal{K}$ is the vector of Riemann constants.

Following Kowalevski, now we will derive the formulae from which one can express P_i,P_{ij} functions in terms of θ functions for hyperelliptic curves.

Let $R(x)$ denote a polynomial in x of degree $2\rho+1$ with real roots

$$R(x)=A_0(x-a_0)(x-a_1)...(x-a_{2\rho})$$

where we assume $a_0>a_1>...>a_{2\rho}$ if $A_0>0$ and if $A_0<0$ then we assume $a_0<a_1<...<a_{2\rho}$. The corresponding hyperelliptic curve of genus ρ is defined by

$$\mathcal{C}:\,y^2=R(x).$$

Suppose that ρ variables $u_1,...u_\rho$ and $x_1,...,x_\rho$ are related by the following equations:

$$\begin{aligned}u_1&=\int_{a_1}^{x_1}\frac{F_1(x)dx}{\sqrt{R(x)}}+\int_{a_3}^{x_2}\frac{F_1(x)dx}{\sqrt{R(x)}}+...+\int_{a_{2\rho-1}}^{x_\rho}\frac{F_1(x)dx}{\sqrt{R(x)}}\\ &...\\ u_\rho&=\int_{a_1}^{x_1}\frac{F_\rho(x)dx}{\sqrt{R(x)}}+\int_{a_3}^{x_2}\frac{F_\rho(x)dx}{\sqrt{R(x)}}+...+\int_{a_{2\rho-1}}^{x_\rho}\frac{F_\rho(x)dx}{\sqrt{R(x)}}.\end{aligned}\tag{115}$$

Here $F_1(x),...,F_\rho(x)$ are polynomials of degree less than ρ. Let us introduce integrals

$$K_{\alpha\beta}=\int_{a_{2\beta-1}}^{a_{2\beta}}\frac{F_\alpha(x)dx}{\sqrt{R(x)}},\quad i\overline{K}_{\alpha\beta}=\int_{a_{2\beta-2}}^{a_{2\beta-1}}\frac{F_\alpha(x)dx}{\sqrt{R(x)}},$$

and

$$iK'_{\mu\nu} = i\overline{K}_{\mu 1} + ... + i\overline{K}_{\mu\nu}.$$

New variables $v_1, ..., v_\rho$ are then defined with equations

$$\begin{aligned} u_1 &= 2K_{11}v_1 + ... + 2K_{1\rho}v_\rho \\ &... \\ u_\rho &= 2K_{\rho 1}v_1 + ... + 2K_{\rho\rho}v_\rho. \end{aligned}$$

Assume that these equations solved in $v_1, ..., v_\rho$ are written in the form

$$\begin{aligned} v_1 &= G_{11}u_1 + ... + G_{1\rho}u_\rho \\ &... \\ v_\rho &= G_{\rho 1}u_1 + ... + G_{\rho\rho}u_\rho. \end{aligned}$$

Kowalevski defines

$$\tau_{\alpha\beta} = 2\imath(G_{1\alpha}K_{1\beta} + ... + G_{\rho\alpha}K_{\rho\beta})$$

and she introduces a function

$$\theta(v_1, ..., v_\rho) = \theta(v|\tau).$$

We are taking some notation from [24]:

$$\theta(v_1, ..., v_\rho)_\lambda = \theta(v + \frac{1}{2}M^\lambda + \frac{1}{2}N^\lambda\tau|\tau), \quad \lambda = 0, 1, ..., 2\rho,$$

where the coordinates of integer vectors $M^\lambda = (m_1^\lambda, ..., m_\rho^\lambda)$ and $N^\lambda = (n_1^\lambda, ..., n_\rho^\lambda)$ are determined with equation

$$\int_\infty^{a_\lambda} \frac{F_\alpha(x)dx}{\sqrt{R(x)}} = m_1^\lambda K_{\alpha 1} + ... + m_\rho^\lambda K_{\alpha\rho} + i(n_1^\lambda K'_{\alpha 1} + ...n_\rho^\lambda K'_{\alpha\rho}).$$

Now, let λ and μ denote two different numbers from $0, 1, ..., 2\rho$. Define vectors of integer numbers $M^\nu = (m_1^\nu, ..., m_\rho^\nu)$, $N^\nu = (n_1^\nu, ..., n_\rho^\nu)$ by next congruences

$$m_\alpha^\nu \equiv m_\alpha^\lambda + m_\alpha^\mu \quad (mod\, 2)$$

$$n_\alpha^\nu \equiv n_\alpha^\lambda + n_\alpha^\mu \quad (mod\, 2)$$

with additional condition that each m_λ^ν is equal 0 or -1 and each n_λ^ν is equal 0 or +1. Finally, denote

$$\theta(v_1, ..., v_\rho)_{\lambda\mu} = \theta(v + \frac{1}{2}M^\nu + \frac{1}{2}N_\nu\tau|\tau).$$

For more informations about these notations and deriving formulae (116)-(118) we refer to [5].

To express the functions P_i, P_{ij}, we use the following formulae:

$$\frac{\sqrt{\varepsilon^\rho(-1)^\alpha\varphi(a_{2\alpha})}}{\sqrt[4]{R'(a_{2\alpha})}} = \frac{\theta(v_1...v_\rho)_{2\alpha}}{\theta(v_1...v_\rho)} \tag{116}$$

$$\frac{\sqrt{\varepsilon^\rho(-1)^{\alpha-1}\varphi(a_{2\alpha-1})}}{\sqrt[4]{R'(a_{2\alpha-1})}} = \frac{\theta(v_1...v_\rho)_{2\alpha-1}}{\theta(v_1...v_\rho)} \tag{117}$$

$$A_0\sqrt{\frac{\pm(a_\lambda-a_\mu)}{A_0}}\sum_{\alpha=1}^{\rho}[\frac{\sqrt{R(x_\alpha)}}{(x_\alpha-a_\lambda)(x_\alpha-a_\mu)\varphi'(x_\alpha)}] = \frac{\theta(v_1...v_\rho)\theta(v_1...v_\rho)_{\lambda\mu}}{\theta(v_1...v_\rho)_\lambda\theta(v_1...v_\rho)_\mu}, \tag{118}$$

where $R(x) = A_0(x-a_0)(x-a_1)...(x-a_{2\rho})$, $\varepsilon = \pm 1$ depending whether $A_0 > 0$ or $A_0 < 0$ and $\phi(x) = (x-x_1)...(x-x_\rho)$.

The system of differential equations that we integrate is of the form of the Abel map for a genus two curve

$$\begin{aligned} 2dt = du_1 &= \frac{s_1\,ds_1}{\sqrt{\Phi(s_1)}} + \frac{s_2\,ds_2}{\sqrt{\Phi(s_2)}} \\ 0 = du_2 &= \frac{ds_1}{\sqrt{\Phi(s_1)}} + \frac{ds_2}{\sqrt{\Phi(s_2)}}. \end{aligned}$$

Then v_1, v_2 are linear functions of time, the polynomial $R(x)$ is represented here by a polynomial $\Phi(s)$ of degree five, and $\phi(s) = (s-s_1)(s-s_2)$.

Next, we use the following relations connecting ratios of the even theta functions of genus two for zero values of arguments, the theta-constants, derived in [5]:

$$\begin{aligned} &c_5^4 : c_0^4 : c_2^4 : c_4^4 : c_{12}^4 : c_{14}^4 : c_{23}^4 : c_{34}^4 : c_{03}^4 : c_{01}^4 = \\ &(a_2-a_4)(a_4-a_0)(a_0-a_2)(a_1-a_3) : (a_1-a_3)(a_3-a_0)(a_0-a_1)(a_2-a_4) \\ &:(a_1-a_3)(a_3-a_2)(a_2-a_1)(a_4-a_0) : (a_1-a_3)(a_3-a_4)(a_4-a_1)(a_2-a_0) \\ &:(a_4-a_0)(a_0-a_1)(a_1-a_4)(a_2-a_3) : (a_0-a_2)(a_2-a_1)(a_1-a_0)(a_4-a_3) \\ &:(a_0-a_4)(a_4-a_3)(a_3-a_0)(a_2-a_1) : (a_0-a_2)(a_2-a_3)(a_3-a_0)(a_4-a_1) \\ &:(a_2-a_4)(a_4-a_3)(a_3-a_2)(a_1-a_0) : (a_2-a_4)(a_4-a_1)(a_1-a_2)(a_3-a_0). \end{aligned} \tag{119}$$

Denote

$$C = (a_3-a_2)^{\frac{3}{2}}\frac{c_{01}c_{03}c_{12}c_{23}c_{14}c_{34}}{c_0^2c_2^2c_4^2}.$$

Finally, we can formulate the relations:

$$\sqrt{(s_1-a_0)(s_2-a_0)} = \frac{C}{\sqrt{a_3-a_1}}\frac{c_5c_0}{c_{01}c_{03}}\frac{\theta_0(v_1,v_2)}{\theta_5(v_1,v_2)},$$
$$\sqrt{(s_1-a_1)(s_2-a_1)} = \frac{C}{\sqrt{a_3-a_1}}\frac{c_0c_2c_4}{c_{03}c_{23}c_{34}}\frac{\theta_1(v_1,v_2)}{\theta_5(v_1,v_2)},$$
$$\sqrt{(s_1-a_2)(s_2-a_2)} = -\imath\frac{C}{\sqrt{a_3-a_1}}\frac{c_5c_2}{c_{12}c_{23}}\frac{\theta_2(v_1,v_2)}{\theta_5(v_1,v_2)},$$
$$\sqrt{(s_1-a_3)(s_2-a_3)} = -\imath\frac{C}{\sqrt{a_3-a_1}}\frac{c_0c_2c_4}{c_{01}c_{12}c_{14}}\frac{\theta_3(v_1,v_2)}{\theta_5(v_1,v_2)},$$
$$\sqrt{(s_1-a_4)(s_2-a_4)} = -\frac{C}{\sqrt{a_3-a_1}}\frac{c_5c_4}{c_{14}c_{34}}\frac{\theta_4(v_1,v_2)}{\theta_5(v_1,v_2)},$$

$$\frac{\sqrt{(s_1-a_2)(s_1-a_3)(s_1-a_4)(s_2-a_0)(s_2-a_1)}}{s_1-s_2} - \frac{\sqrt{(s_1-a_0)(s_1-a_1)(s_2-a_2)(s_2-a_3)(s_2-a_4)}}{s_1-s_2} = -\imath C\frac{c_5}{c_{03}}\frac{\theta_{01}(v_1,v_2)}{\theta_5(v_1,v_2)},$$

$$\frac{\sqrt{(s_1-a_1)(s_1-a_3)(s_1-a_4)(s_2-a_0)(s_2-a_2)}}{s_1-s_2} - \frac{\sqrt{(s_1-a_0)(s_1-a_2)(s_2-a_1)(s_2-a_3)(s_2-a_4)}}{s_1-s_2} = C\frac{c_5}{c_4}\frac{\theta_{02}(v_1,v_2)}{\theta_5(v_1,v_2)},$$

$$\frac{\sqrt{(s_1-a_1)(s_1-a_2)(s_1-a_4)(s_2-a_0)(s_2-a_3)}}{s_1-s_2} - \frac{\sqrt{(s_1-a_0)(s_1-a_3)(s_2-a_1)(s_2-a_2)(s_2-a_4)}}{s_1-s_2} = -C\frac{c_5}{c_{01}}\frac{\theta_{03}(v_1,v_2)}{\theta_5(v_1,v_2)},$$

$$\frac{\sqrt{(s_1-a_1)(s_1-a_2)(s_1-a_3)(s_2-a_0)(s_2-a_4)}}{s_1-s_2} - \frac{\sqrt{(s_1-a_0)(s_1-a_4)(s_2-a_1)(s_2-a_2)(s_2-a_3)}}{s_1-s_2} = -\imath C\frac{c_5}{c_2}\frac{\theta_{04}(v_1,v_2)}{\theta_5(v_1,v_2)},$$

$$\frac{\sqrt{(s_1-a_0)(s_1-a_3)(s_1-a_4)(s_2-a_1)(s_2-a_2)}}{s_1-s_2} - \frac{\sqrt{(s_1-a_1)(s_1-a_2)(s_2-a_0)(s_2-a_3)(s_2-a_4)}}{s_1-s_2} = C\frac{c_5}{c_{23}}\frac{\theta_{12}(v_1,v_2)}{\theta_5(v_1,v_2)},$$

$$\frac{\sqrt{(s_1-a_0)(s_1-a_2)(s_1-a_4)(s_2-a_1)(s_2-a_3)}}{s_1-s_2} - \frac{\sqrt{(s_1-a_1)(s_1-a_3)(s_2-a_0)(s_2-a_2)(s_2-a_4)}}{s_1-s_2} = -C\frac{\theta_{13}(v_1,v_2)}{\theta_5(v_1,v_2)},$$

$$\frac{\sqrt{(s_1-a_0)(s_1-a_2)(s_1-a_3)(s_2-a_1)(s_2-a_4)}}{s_1-s_2}$$
$$-\frac{\sqrt{(s_1-a_1)(s_1-a_4)(s_2-a_0)(s_2-a_2)(s_2-a_3)}}{s_1-s_2}=-\imath C\frac{c_5}{c_{34}}\frac{\theta_{14}(v_1,v_2)}{\theta_5(v_1,v_2)}.$$

$$\frac{\sqrt{(s_1-a_0)(s_1-a_1)(s_1-a_4)(s_2-a_2)(s_2-a_3)}}{s_1-s_2}$$
$$-\frac{\sqrt{(s_1-a_2)(s_1-a_3)(s_2-a_0)(s_2-a_1)(s_2-a_4)}}{s_1-s_2}=\imath C\frac{c_5}{c_{12}}\frac{\theta_{23}(v_1,v_2)}{\theta_5(v_1,v_2)},$$

$$\frac{\sqrt{(s_1-a_0)(s_1-a_1)(s_1-a_3)(s_2-a_2)(s_2-a_4)}}{s_1-s_2}$$
$$-\frac{\sqrt{(s_1-a_2)(s_1-a_4)(s_2-a_0)(s_2-a_1)(s_2-a_3)}}{s_1-s_2}=-C\frac{c_5}{c_0}\frac{\theta_{24}(v_1,v_2)}{\theta_5(v_1,v_2)},$$

$$\frac{\sqrt{(s_1-a_0)(s_1-a_1)(s_1-a_2)(s_2-a_3)(s_2-a_4)}}{s_1-s_2}$$
$$-\frac{\sqrt{(s_1-a_3)(s_1-a_4)(s_2-a_0)(s_2-a_1)(s_2-a_2)}}{s_1-s_2}=-C\frac{c_5}{c_{14}}\frac{\theta_{34}(v_1,v_2)}{\theta_5(v_1,v_2)}.$$

We will demonstrate how these relations are obtained on two examples. First, for $\sqrt{(s_1-a_3)(s_2-a_3)}$: from (117), it follows that

$$\sqrt{(s_1-a_3)(s_2-a_3)}=\sqrt{\varphi(a_3)}=-\imath\sqrt[4]{\Phi'(a_3)}\frac{\theta_3(v_1,v_2)}{\theta_5(v_1,v_2)}.$$

Using the proportion (119) we get

$$\begin{aligned}
&\frac{C}{\sqrt{a_3-a_1}}\frac{c_0c_2c_4}{c_{01}c_{12}c_{14}}=(a_3-a_1)\frac{c_{03}c_{23}c_{34}}{c_0c_2c_4}=\\
&=(a_3-a_1)\left(\frac{(a_4-a_3)(a_2-a_3)(a_3-a_0)}{(a_1-a_3)^3}\right)^{\frac14}\\
&=\sqrt[4]{(a_3-a_0)(a_3-a_1)(a_3-a_2)(a_3-a_4)}=-\frac{\sqrt[4]{\Phi'(a_3)}}{4}.
\end{aligned}$$

As the second example, let us derive one of the formula for P_{ij}. From previous expressions for $\sqrt{(s_1-a_i)(s_1-a_j)}$ we have:

$$
\begin{aligned}
&\frac{\theta_5(v_1,v_2)}{\theta_2(v_1,v_2)}\frac{\theta_5(v_1,v_2)}{\theta_3(v_1,v_2)}\\
&=\frac{C^2}{a_3-a_1}\frac{c_0c_2^2c_4c_5}{c_{01}c_{12}^2c_{14}c_{23}}\frac{1}{\sqrt{(s_1-a_2)(s_2-a_2)}\sqrt{(s_1-a_3)(s_2-a_3)}}\\
&=C\sqrt{a_3-a_1}\frac{c_{03}c_{34}c_5}{c_0c_4c_{12}}\frac{1}{\sqrt{(s_1-a_2)(s_2-a_2)}\sqrt{(s_1-a_3)(s_2-a_3)}}
\end{aligned}
$$

and from the proportion (119)

$$
\frac{c_{03}c_{34}}{c_0c_4}=\sqrt{\frac{a_2-a_3}{a_3-a_1}}
$$

we get

$$
\frac{\theta_5(v_1,v_2)\theta_5(v_1,v_2)}{\theta_2(v_1,v_2)\theta_3(v_1,v_2)}=C\sqrt{a_2-a_3}\frac{c_5}{c_{12}}\frac{1}{\sqrt{(s_1-a_2)(s_2-a_2)}\sqrt{(s_1-a_3)(s_2-a_3)}}.
$$

Finally, using relation (118), we get

$$
\begin{aligned}
&\frac{\sqrt{(s_1-a_0)(s_1-a_1)(s_2-a_2)(s_2-a_3)(s_1-a_4)}}{s_1-s_2}\\
&-\frac{\sqrt{(s_2-a_0)(s_2-a_1)(s_2-a_2)(s_1-a_3)(s_2-a_4)}}{s_1-s_2}\\
&=\frac{\theta_{23}(v_1,v_2)}{\theta_5(v_1,v_2)}\frac{\theta_5(v_1,v_2)}{\theta_2(v_1,v_2)}\frac{\theta_5(v_1,v_2)}{\theta_3(v_1,v_2)}\frac{\sqrt{(s_1-a_2)(s_2-a_2)}\sqrt{(s_1-a_3)(s_2-a_3)}}{\sqrt{\pm(a_2-a_3)}}\\
&=\imath C\frac{c_5}{c_{12}}\frac{\theta_{23}(v_1,v_2)}{\theta_5(v_1,v_2)}.
\end{aligned}
$$

Notice here that the zeros $a_0,...,a_4$ of the polynomial $R=\Phi$ in the cases of systems of Kowalevski type mentioned in previous Sections, were denoted by $l_1,l_2,l_3,k,-k$. By P_i we denoted $\sqrt{(s_1-l_i)(s_2-l_i)}$ for $i=1,2,3$. The expressions in terms of theta-functions depend on the order of the roots $l_1,l_2,l_3,k,-k$, since we assume from the beginning of this Appendix that the zeros $a_0,...,a_4$ are numerated according to their order.

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